

ON THE CAHN-HILLIARD-BRINKMAN SYSTEM

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ABSTRACT. We consider a diffuse interface model for phase separation of an isothermal incompressible binary fluid in a Brinkman porous medium. The coupled system consists of a convective Cahn-Hilliard equation for the phase field ϕ , i.e., the difference of the (relative) concentrations of the two phases, coupled with a modified Darcy equation proposed by H.C. Brinkman in 1947 for the fluid velocity \mathbf{u} . This equation incorporates a diffuse interface surface force proportional to $\phi \nabla \mu$, where μ is the so-called chemical potential. We analyze the well-posedness of the resulting Cahn-Hilliard-Brinkman (CHB) system for (ϕ, \mathbf{u}) . Then we establish the existence of a global attractor and the convergence of a given (weak) solution to a single equilibrium via Łojasiewicz-Simon inequality. Furthermore, we study the behavior of the solutions as the viscosity goes to zero, that is, when the CHB system approaches the Cahn-Hilliard-Hele-Shaw (CHHS) system. We first prove the existence of a weak solution to the CHHS system as limit of CHB solutions. Then, in dimension two, we estimate the difference of the solutions to CHB and CHHS systems in terms of the viscosity constant appearing in CHB.

Keywords: Incompressible binary fluids, Brinkman equation, Darcy's law, diffuse interface models, Cahn-Hilliard equation, weak solutions, existence, uniqueness, global attractor, convergence to equilibrium, vanishing viscosity.

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1. INTRODUCTION

The so-called Brinkman equation was proposed by H.C. Brinkman in [6] as a modified Darcy's law in order to describe the flow through a porous mass. If we assume that the incompressible fluid occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, for any time $t \in (0, T)$, $T > 0$, the Brinkman equation for the (divergence free) fluid velocity \mathbf{u} reads

$$-\nabla \cdot [\nu D(\mathbf{u})] + \eta \mathbf{u} = -\nabla p,$$

in $\Omega \times (0, T)$. Here $2D(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^{tr}$, $\nu > 0$ is the viscosity, $\eta > 0$ the fluid permeability and p is the fluid pressure.

More recently, a diffuse interface variant of Brinkman equation has been proposed to model phase separation of incompressible binary fluids in a porous medium (see [21]). Let us suppose that both the fluids have equal constant density and indicate by ϕ the difference of the fluid (relative) concentrations. Denoting by \mathbf{u} the (averaged) fluid velocity, the

resulting model is the following

$$(1.1) \quad \partial_t \phi + \nabla \cdot (\phi \mathbf{u}) = \nabla \cdot (M \nabla \mu),$$

$$(1.2) \quad \mu = -\varepsilon \Delta \phi + \frac{1}{\varepsilon} f(\phi),$$

$$(1.3) \quad -\nabla \cdot [\nu D(\mathbf{u})] + \eta \mathbf{u} = -\nabla p - \gamma \phi \nabla \mu,$$

$$(1.4) \quad \nabla \cdot \mathbf{u} = 0,$$

in $\Omega \times (0, T)$. Here $M > 0$ stands for the mobility, $\varepsilon > 0$ is related to the diffuse interface thickness, f is the derivative of a double well potential describing phase separation, and $\gamma > 0$ is a surface tension parameter.

This model consists of a convective Cahn-Hilliard equation (1.1)-(1.2) coupled with the Brinkman equation through the surface tension force $\gamma \phi \nabla \mu$. For this reason (1.1)-(1.4) has been called Cahn-Hilliard-Brinkman (CHB) system. Such a system belongs to a class of diffuse interface models which are used to describe the behavior of multi-phase fluids. We recall, in particular, the Cahn-Hilliard-Navier-Stokes system which has been investigated in several papers (see, e.g., [2, 3, 4, 7, 11, 12, 13, 19, 23, 30, 31], cf. also [15] for a recent review on modeling and numerics).

CHB system has recently been analyzed from the numerical viewpoint in [9]. More precisely, the authors have considered system (1.1)-(1.4) with M , ν and η possibly depending on ϕ and endowed with the boundary and initial conditions

$$(1.5) \quad \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \text{on } \partial\Omega \times (0, T),$$

$$(1.6) \quad \partial_{\mathbf{n}} \phi = \partial_{\mathbf{n}} \mu = 0, \quad \text{on } \partial\Omega \times (0, T),$$

$$(1.7) \quad \phi(0) = \phi_0,$$

where $\phi_0 : \Omega \rightarrow \mathbb{R}$ is a given function. Here \mathbf{n} stands for the outward normal vector to $\partial\Omega$ which is supposed to be smooth enough.

The main goal of this contribution is to establish some theoretical results on (1.1)-(1.7), in the case when M , ν and η are constant. First of all we analyze the well-posedness of the problem, proving the global existence and uniqueness of a weak solution and its continuous dependence on the initial datum. Secondly, we study the longterm behavior of the CHB system as a dissipative dynamical system by proving the existence of a global attractor. Then we investigate the long-time dynamics of any given weak solution by showing that each trajectory does converge to a unique stationary state, with an explicit convergence rate. Our results includes the case $\eta = 0$ (see [22] and references therein).

In the second part of the paper we analyze the behavior of solutions when ν goes to zero. Observe that when $\nu = 0$ system (1.1)-(1.4) becomes the so-called Cahn-Hilliard-Hele-Shaw (CHHS) model. This is a particularly challenging problem which finds applications in tumor growth dynamics (see, e.g., [20] and its references) and has been recently studied from the theoretical viewpoint in [20, 27, 28] (see also [10, 16, 17] and references therein).

We are able to prove that there is a global weak solution to CHHS system which is the limit of solutions to CHB system with (1.5)-(1.7) (compare with [10, Thm.2.4]). Notice that uniqueness of weak solutions is still an open problem. On the contrary, a *strong* solution is unique, but, if $d = 3$, only local existence is known so far unless the initial datum is a small perturbation of a suitable constant state (see [20]).

In dimension two, we also provide an estimate of the difference of (strong) solutions to CHB and CHHS systems with respect to ν .

The plan of this paper goes as follows. In the next section we state the main results along with some notation and basic tools. Section 3 is devoted to prove certain a priori estimates. Then, in Section 4, we establish the well-posedness of problem (1.1)-(1.7) and a global dissipative estimate. In Section 5 we obtain some higher-order estimates which are helpful to prove the existence of the global attractor as well as to show, in Section 6, the convergence to the equilibrium of a given weak solution. Finally, in Section 7, we analyze what happens when ν goes to zero, while in Section 8 we estimate the difference of (strong) solutions to CHB and CHHS systems.

2. PRELIMINARIES AND MAIN RESULTS

Here we list our assumptions on f and the potential $F(s) := \int_0^s f(y) dy$ and we introduce some notation. Then we state our main results. This requires to formulate our problems rigorously. We also recall a pair of Gronwall-type lemmas.

Assumptions on F and f . We assume that $f \in C^1(\mathbb{R})$, with $f(0) = 0$, is such that

$$(2.1) \quad |f(s)| \leq c(1 + |s|^3),$$

and

$$(2.2) \quad F(s) \geq -c,$$

for all $s \in \mathbb{R}$ and some $c > 0$. In the course of the investigation we shall need further assumptions such as

$$(2.3) \quad |f'(s) - f'(t)| \leq c|s - t|(1 + |s| + |t|),$$

or the stronger condition $f \in C^2(\mathbb{R})$ such that

$$(2.4) \quad |f''(s)| \leq c(1 + |s|).$$

We shall also make use of the following dissipation condition

$$(2.5) \quad \inf_{s \in \mathbb{R}} f'(s) > -\infty.$$

A typical example of (regular) double well potential is

$$(2.6) \quad F(s) = (s^2 - 1)^2,$$

which complies with (2.1)-(2.5). More generally, one can take a fourth degree polynomial with positive leading coefficient.

Functional spaces. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be either a smooth bounded connected domain or a convex polygonal or polyhedral domain. For any positive integer r , let $H^r(\Omega) = W^{r,2}(\Omega)$, the usual Sobolev space, and denote the norm $\|\cdot\|_{W^{r,2}(\Omega)}$ by $\|\cdot\|_r$. Throughout the paper, we set $H = L^2(\Omega)$,

$$V = \overline{\{\phi \in C^\infty(\overline{\Omega}) : \partial_n \phi = 0 \text{ on } \partial\Omega\}}^{H^1(\Omega)} \quad \text{and} \quad H^r = H^r(\Omega) \cap V,$$

endowed with the norm $\|\cdot\|_r$. Similarly, we denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$. The shorthand $\langle \cdot, \cdot \rangle$ will stand both for the scalar product in H and for the duality product between H^r

and its dual space H^{-r} . The same symbols will also be used for the scalar product and norm in spaces of vector-valued elements.

Besides, let \mathcal{V} be the space of divergence-free test functions defined by

$$\mathcal{V} = \{\mathbf{v} \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^3) : \nabla \cdot \mathbf{v} = 0\}.$$

We shall use the following spaces

$$\mathbf{H} = \overline{\mathcal{V}}^{(H)^3} \quad \text{and} \quad \mathbf{V} = \overline{\mathcal{V}}^{(H^1)^3}.$$

In particular we recall that if $\mathbf{v} \in \mathbf{V}$ then $\mathbf{v}|_{\partial\Omega} = \mathbf{0}$ and if $\mathbf{v} \in \mathbf{H}$ then $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ (see, e.g., [26, Chapter I]).

Notation. Without loss of generality we will set $M = \varepsilon = \gamma = 1$. Throughout the paper, $c \geq 0$ will stand for a generic constant and $\mathcal{Q}(\cdot)$ for a generic positive increasing function.

2.1. Statement of the main results. Let us introduce the definition of weak solution to the CHB system with boundary and initial conditions (1.5)-(1.7).

Definition 2.1. Let $\nu > 0$, $\phi_0 \in H^1$ and $T > 0$ be given. A pair (ϕ, \mathbf{u}) is a (weak) solution to system (1.1)-(1.4) endowed with (1.5)-(1.7) if

$$\phi \in \mathcal{C}([0, T], H^1) \cap L^2(0, T; H^3)$$

satisfies

$$\begin{aligned} (2.7) \quad & \langle \partial_t \phi(t), w \rangle + \langle \nabla \cdot (\phi(t) \mathbf{u}(t)), w \rangle + \langle \nabla \mu(t), \nabla w \rangle = 0, \quad \forall w \in H^1, \quad \text{a.e. } t \in [0, T], \\ & \partial_{\mathbf{n}} \phi = 0, \quad \text{a.e. on } \partial\Omega \times (0, T), \\ & \phi|_{t=0} = \phi_0, \quad \text{a.e. in } \Omega, \end{aligned}$$

with $\mu \in L^2(0, T; H^1)$ given by (1.2) and

$$\mathbf{u} \in L^2(0, T; \mathbf{V})$$

fulfills

$$(2.8) \quad \nu \langle \nabla \mathbf{u}(t), \nabla \mathbf{v} \rangle + \eta \langle \mathbf{u}(t), \mathbf{v} \rangle = - \langle \phi(t) \nabla \mu(t), \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}, \quad \text{a.e. } t \in [0, T].$$

Remark 2.2. It is straightforward to observe that any weak solution satisfies mass conservation, namely,

$$(2.9) \quad \langle \phi(t) \rangle = \langle \phi_0 \rangle, \quad \forall t \geq 0,$$

where

$$\langle \phi(t) \rangle := \frac{1}{|\Omega|} \int_{\Omega} \phi(\mathbf{x}, t) \, d\mathbf{x}.$$

Remark 2.3. As we shall see in Section 3, the regularity assumed in Definition 2.1 yields

$$\nabla \cdot (\phi \mathbf{u}) \in L^2(0, T; H^{-1}),$$

so that $\partial_t \phi \in L^2(0, T; H^{-1})$ by comparison. Besides, we have

$$\phi \nabla \mu \in L^{8/5}(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}^*).$$

Remark 2.4. As usual the pressure term is dropped in the weak formulation of the Stokes problem. Indeed, the pressure can be recovered (up to a constant) thanks to a classical result (see, for instance, [26, Theorem I.1.4]). In particular, since

$$\mathcal{S} = \nu \Delta \mathbf{u} - \eta \mathbf{u} + \phi \nabla \mu \in L^2(0, T; \mathbf{V}^*),$$

we know that there exists a unique (up to an additive constant) function

$$p \in L^2(0, T; H)$$

satisfying $\nabla p = \mathcal{S}$.

Global existence and uniqueness of a weak solution is given by

Theorem 2.5. *Let $\nu > 0$, $\eta \geq 0$ and f satisfy (2.1)-(2.2). Let $\phi_0 \in H^1$ be given. Then, for every $T > 0$, there exists a pair (ϕ, \mathbf{u}) which is a solution to the CHB system according to Definition 2.1. If (2.3) holds, then the weak solution is unique.*

We also have continuous dependence estimates.

Theorem 2.6. *Let $\nu > 0$, $\eta > 0$. Under the same assumptions of Theorem 2.5, if (ϕ_1, \mathbf{u}_1) and (ϕ_2, \mathbf{u}_2) are two weak solutions to the CHB system such that $\langle \phi_1(0) \rangle = \langle \phi_2(0) \rangle$, then, for every $T > 0$, there exists $C_T > 0$ depending on $R = \max\{\|\phi_1(0)\|_1, \|\phi_2(0)\|_1\}$ such that the following continuous dependence estimates hold*

$$(2.10) \quad \|\phi_1(t) - \phi_2(t)\|_1^2 \leq \|\phi_1(0) - \phi_2(0)\|_1^2 e^{C_T/\sqrt{\nu}},$$

and

$$(2.11) \quad \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_1^2 ds \leq \|\phi_1(0) - \phi_2(0)\|_1^2 \left(1 + C_T e^{C_T/\sqrt{\nu}}\right),$$

for every $t \in [0, T]$.

Remark 2.7. In the case $\eta = 0$ (cf. [22]) the same continuous dependence estimates hold by replacing $\sqrt{\nu}$ with ν in (2.10) and (2.11).

The next result shows that any weak solution converges to a single stationary state as time goes to infinity.

Theorem 2.8. *Let $\nu > 0$, $\eta \geq 0$ and let f be real analytic satisfying (2.2)-(2.5). For every fixed $\phi_0 \in H^1$, the global solution ϕ originating from ϕ_0 converges to an equilibrium ϕ^* as $t \rightarrow \infty$, with the following convergence rate*

$$(2.12) \quad \|\phi(t) - \phi^*\|_1 \leq \frac{c}{(1+t)^{\theta/(1-2\theta)}}, \quad \forall t \geq t^*,$$

for some $\theta = \theta(\phi^*) \in (0, \frac{1}{2})$, $c = c(\|\phi_0\|_1) \geq 0$ and $t^* > 0$. Here $\phi^* \in H^2$ is a solution to the stationary system

$$-\Delta z + f(z) = \text{const in } \Omega, \quad \partial_{\mathbf{n}} z = 0 \text{ on } \partial\Omega, \quad \langle z \rangle = \langle \phi_0 \rangle.$$

Furthermore, the velocity field \mathbf{u} vanishes and satisfies

$$(2.13) \quad \|\mathbf{u}(t)\|_1 \leq \frac{c_\nu}{(1+t)^{\theta/4(1-2\theta)}}, \quad \forall t \geq t^*,$$

where $c_\nu \rightarrow \infty$ as $\nu \rightarrow 0$.

Let us now introduce the definition of weak solution to the CHHS system endowed with (1.6)-(1.7) and

$$(2.14) \quad \mathbf{u} \cdot \mathbf{n} = \mathbf{0}, \quad \text{on } \partial\Omega \times (0, T).$$

Definition 2.9. Let $\phi_0 \in H^1$ and $T > 0$ be given. A pair (ϕ, \mathbf{u}) is a (weak) solution to the CHHS system endowed with (1.6)-(1.7) and (2.14) if

$$\phi \in \mathcal{C}_w([0, T], H^1) \cap L^2(0, T; H^3)$$

satisfies

$$\begin{aligned} \langle \partial_t \phi(t), w \rangle + \langle \nabla \cdot (\phi(t) \mathbf{u}(t)), w \rangle + \langle \nabla \mu(t), \nabla w \rangle &= 0, \quad \forall w \in H^1, \quad \text{a.e. } t \in [0, T], \\ \partial_{\mathbf{n}} \phi &= 0, \quad \text{a.e. on } \partial\Omega \times (0, T), \\ \phi|_{t=0} &= \phi_0, \quad \text{a.e. in } \Omega, \end{aligned}$$

with $\mu \in L^2(0, T; H^1)$ given by (1.2) and

$$\mathbf{u} \in L^2(0, T; \mathbf{H})$$

fulfills

$$\eta \langle \mathbf{u}(t), \mathbf{v} \rangle = -\langle \phi(t) \nabla \mu(t), \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}, \quad \text{a.e. } t \in [0, T].$$

Remark 2.10. It is worth noting that the regularity assumed in Definition 2.9 yields

$$\nabla \cdot (\phi \mathbf{u}) \in L^{8/5}(0, T; H^{-1}) \quad \text{whence} \quad \partial_t \phi \in L^{8/5}(0, T; H^{-1}).$$

The following theorem says that a weak solution to the CHHS system can be found as a limit of solutions to CHB system as viscosity vanishes.

Theorem 2.11. Let $\eta > 0$ and let f satisfy (2.1)-(2.2). For $\phi_0 \in H^1$ let $\{\nu_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\nu_n \rightarrow 0$ as $n \rightarrow \infty$. Let (ϕ_n, \mathbf{u}_n) be the sequence of weak solutions corresponding to the CHB system with $\nu = \nu_n$ originating from ϕ_0 . Then, up to a subsequence, (ϕ_n, \mathbf{u}_n) converges to a weak solution (ϕ, \mathbf{u}) to the CHHS system according to Definition 2.9 in the following sense:

$$\begin{aligned} \phi_n &\rightarrow \phi \quad \text{weakly in } L^2(0, T; H^3) \text{ and strongly in } L^2(0, T; H^2), \\ \mathbf{u}_n &\rightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; \mathbf{H}). \end{aligned}$$

Finally, in dimension two, we state a result about the estimate of the difference between a solution to the BCH system and a solution to the CHHS system. Indeed, it is known from [20] that the CHHS system endowed with (1.6)-(1.7) and (2.14) admits a unique *strong* solution provided that $\phi_0 \in H^2$, which is also *global* when $d = 2$. In this case, we have the following result

Theorem 2.12. Let $d = 2$ and $\eta > 0$. Let f satisfy (2.2)-(2.3). Take $\phi_0^\nu, \phi_0 \in H^2$ such that $\langle \phi_0^\nu \rangle = \langle \phi_0 \rangle$ and set

$$R := \sup_{\nu > 0} \{ \|\phi_0^\nu\|_2, \|\phi_0\|_2 \} < \infty.$$

Let $(\phi_\nu, \mathbf{u}_\nu)$ be the unique weak solution to the CHB system with $\nu > 0$, originating from ϕ_0^ν , and (ϕ, \mathbf{u}) the solution to the CHHS system with initial datum ϕ_0 . Then, for every

$T > 0$, there exists $C_T > 0$ (depending only on R) such that

$$\|\phi_\nu(t) - \phi(t)\|_1^2 + \int_0^t \|\mathbf{u}_\nu(y) - \mathbf{u}(y)\|^2 dy \leq \|\phi_0^\nu - \phi_0\|_1^2 e^{C_T} + C_T \nu, \quad \forall t \in [0, T].$$

In particular, if $\phi_0^\nu = \phi_0$, then

$$\phi_\nu \rightarrow \phi \quad \text{in } L^\infty(0, T; H^1) \quad \text{as } \nu \rightarrow 0,$$

for all $T > 0$.

2.2. Basic inequalities. We will exploit the classical inequalities due to Sobolev, Gagliardo and Nirenberg, Agmon and Poincaré, respectively, which are standard (see, e.g., [24, 26]).

We also need a pair of Gronwall-type inequalities. The uniform Gronwall lemma ([25, Section 1.1.3]), namely,

Lemma 2.13. *Let ψ_0 be an absolutely continuous nonnegative function and ψ_1, ψ_2 be two nonnegative functions satisfying, almost everywhere in \mathbb{R}^+ , the differential inequality*

$$\frac{d}{dt} \psi_0 \leq \psi_0 \psi_1 + \psi_2.$$

Assume also that

$$\sup_{t \geq 0} \int_t^{t+r} \psi_i(\tau) d\tau \leq m_i, \quad i = 0, 1, 2,$$

for some positive constants m_i and $r > 0$. Then,

$$\psi_0(t+r) \leq \left(\frac{m_0}{r} + m_2 \right) e^{m_1}, \quad \forall t \geq 0.$$

The following differential Gronwall lemma whose proof is reported here below.

Lemma 2.14. *Let $\psi : [t^*, \infty) \rightarrow \mathbb{R}$ be an absolutely continuous function, which fulfills for almost every $t \geq t^*$ the differential inequality*

$$\frac{d}{dt} \psi(t) + \alpha \psi(t) \leq (1+t)^{-\beta},$$

for some $\alpha > 0$ and $\beta > 0$. Then, there exists $c > 0$ such that, for every sufficiently large time t

$$\psi(t) \leq c(1 + \psi(t^*))(1+t)^{-\beta}.$$

Proof. Multiplying the above inequality by $e^{\alpha t}$ we deduce

$$\frac{d}{dt} (e^{\alpha t} \psi(t)) \leq (1+t)^{-\beta} e^{\alpha t}.$$

Integrating this inequality between t^* and t we obtain

$$e^{\alpha t} \psi(t) \leq e^{\alpha t^*} \psi(t^*) + \int_{t^*}^t (1+s)^{-\beta} e^{\alpha s} ds.$$

We now estimate the integral appearing on the right hand side of this last inequality. For $t \geq 2t^*$, we have

$$\begin{aligned} \int_{t^*}^t (1+s)^{-\beta} e^{\alpha s} ds &= \int_{t^*}^{\frac{t}{2}} e^{\alpha s} (1+s)^{-\beta} ds + \int_{\frac{t}{2}}^t (1+s)^{-\beta} e^{\alpha s} ds \\ &\leq e^{\alpha \frac{t}{2}} \int_{t^*}^{\frac{t}{2}} (1+s)^{-\beta} ds + \left(1 + \frac{t}{2}\right)^{-\beta} \int_{\frac{t}{2}}^t e^{\alpha s} ds, \end{aligned}$$

hence we deduce

$$e^{-\alpha t} \int_{t^*}^t (1+s)^{-\beta} e^{\alpha s} ds \leq e^{-\alpha \frac{t}{2}} \int_{t^*}^{\frac{t}{2}} (1+s)^{-\beta} ds + c(1+t)^{-\beta}.$$

This yields

$$\psi(t) \leq e^{-\alpha(t-t^*)} \psi(t^*) + ce^{-\alpha \frac{t}{2}} (1+t)^{-\beta+1} + c(1+t)^{-\beta}$$

which, for sufficiently large times, reduces to the claimed inequality. \square

3. BASIC ESTIMATES

In this section we let $\phi_0 \in H^1$ and we denote by (ϕ, \mathbf{u}) a weak solution to the CHB system originating from ϕ_0 . Our aim is to prove a number of a priori estimates for (ϕ, \mathbf{u}) .

To this aim, in the following we denote by $\mathcal{Q}(\cdot)$ a generic increasing and positive function which is *independent of* ν . All the energy estimates are formal but they can be performed rigorously within a Galerkin approximation scheme (see Subsection 4.1).

3.1. Energy estimates.

Lemma 3.1. *For any given $R > 0$ the following inequality holds*

$$(3.1) \quad \|\phi(t)\|_1^2 + \int_0^\infty (\|\nabla \mu(y)\|^2 + \eta \|\mathbf{u}(y)\|^2) dy + \nu \int_0^\infty \|\nabla \mathbf{u}(y)\|^2 dy \leq \mathcal{Q}(R),$$

for every initial datum ϕ_0 with $\|\phi_0\|_1 \leq R$. Besides, for every $T > 0$, we have

$$(3.2) \quad \int_0^T (\|\mu(y)\|_1^2 + \|\phi(y)\|_3^2) dy \leq \mathcal{Q}_T(R),$$

for some increasing positive function \mathcal{Q}_T depending on T .

Proof. Taking $w = \mu$ in (2.7) and $\mathbf{v} = \mathbf{u}$ in (2.8), and summing up the resulting equalities, we have

$$(3.3) \quad \frac{d}{dt} \left(\frac{1}{2} \|\nabla \phi\|^2 + \langle F(\phi), 1 \rangle \right) + \|\nabla \mu\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \eta \|\mathbf{u}\|^2 = 0.$$

In light of (2.1), this provides

$$\|\phi(t)\|_1 \leq \|\phi_0\|_1 + 2\langle F(\phi_0), 1 \rangle \leq \mathcal{Q}(R), \quad \forall t \geq 0.$$

A subsequent integration in time of (3.3) completes the proof of (3.1).

Now, multiplying (1.2) in H by the constant function 1, we get

$$\langle \mu, 1 \rangle = \langle f(\phi), 1 \rangle,$$

which, by (2.1), gives

$$\langle \mu \rangle \leq c(1 + \int_{\Omega} |\phi|^3) \leq c(1 + \|\phi\|_1^3) \leq \mathcal{Q}(R).$$

Thanks to (3.1) we obtain, for every $T > 0$,

$$\int_0^T \|\mu(y)\|_1^2 dy \leq \mathcal{Q}_T(R),$$

hence $\mu \in L^2(0, T; H^1)$. Let us now multiply (1.2) by $-\Delta^2 \phi$ in H . This yields

$$\langle \nabla \mu, \nabla \Delta \phi \rangle = -\|\nabla \Delta \phi\|^2 + \langle f'(\phi) \nabla \phi, \nabla \Delta \phi \rangle.$$

On the other hand, recalling (2.1) and (3.1), we have

$$\begin{aligned} \langle f'(\phi) \nabla \phi, \nabla \Delta \phi \rangle &\leq \|f'(\phi)\|_{L^3} \|\nabla \phi\|_{L^6} \|\nabla \Delta \phi\| \\ &\leq \mathcal{Q}(R) \|\nabla \phi\|^{1/2} \|\nabla \Delta \phi\|^{1/2} \|\nabla \Delta \phi\| \\ &\leq \mathcal{Q}(R) + \frac{1}{4} \|\nabla \Delta \phi\|^2, \end{aligned}$$

which entails

$$\frac{1}{2} \|\nabla \Delta \phi\|^2 \leq \|\nabla \mu\|^2 + \mathcal{Q}(R).$$

This gives

$$\int_0^T \|\nabla \Delta \phi(y)\|^2 dy \leq \mathcal{Q}(R),$$

and, owing to (3.1), we find

$$(3.4) \quad \int_0^T \|\phi(y)\|_3^2 dy \leq \mathcal{Q}(R),$$

so that $\phi \in L^2(0, T; H^3)$. □

Remark 3.2. Since $\phi \in L^\infty(\mathbb{R}^+; H^1) \cap L^2(0, T; H^3)$, we easily get by interpolation

$$\int_0^T \|\phi\|_2^p dt \leq \int_0^T \|\phi\|_1^{p/2} \|\phi\|_3^{p/2} dt \leq c \int_0^T \|\phi\|_3^{p/2} dt < \infty \quad \text{if } p \leq 4.$$

Thus

$$(3.5) \quad \int_0^T \|\phi(y)\|_2^4 dy \leq \mathcal{Q}(R),$$

that is, $\phi \in L^4(0, T; H^2)$.

3.2. Further Estimates.

The term $\nabla \cdot (\phi \mathbf{u})$. For $w \in H^1$, using the Agmon inequality and interpolation, we compute

$$\begin{aligned} \langle \nabla \cdot (\phi \mathbf{u}), w \rangle &= \langle \phi \mathbf{u}, \nabla w \rangle \leq \|\nabla w\| \|\mathbf{u}\| \|\phi\|_{L^\infty} \\ &\leq \|\nabla w\| \|\mathbf{u}\| \|\phi\|_1^{3/4} \|\phi\|_3^{1/4} \leq \mathcal{Q}(R) \|\nabla w\| \|\mathbf{u}\| \|\phi\|_3^{1/4}. \end{aligned}$$

This implies

$$\begin{aligned} \left| \int_0^T \langle \nabla \cdot (\phi \mathbf{u}), w \rangle dt \right| &\leq \mathcal{Q}(R) \int_0^T \|\nabla w\| \|\mathbf{u}\| \|\phi\|_3^{1/4} dt \\ &\leq \mathcal{Q}(R) \left(\int_0^T \|\nabla w\|^{8/3} dt \right)^{3/8} \left(\int_0^T \|\mathbf{u}\|^2 dt \right)^{1/2} \left(\int_0^T \|\phi\|_3^2 dt \right)^{1/8}. \end{aligned}$$

As a consequence, invoking the fact that $\mathbf{u} \in L^2(0, T; \mathbf{H})$ and $\phi \in L^2(0, T; H^3)$, we get

$$\left| \int_0^T \langle \nabla \cdot (\phi \mathbf{u}), w \rangle dt \right| \leq \mathcal{Q}_T(R) \left(\int_0^T \|\nabla w\|^{8/3} dt \right)^{3/8},$$

which gives

$$\nabla \cdot (\phi \mathbf{u}) \in L^{8/5}(0, T; H^{-1}).$$

We stress that this control is independent of ν . Exploiting the ν -dependent estimate $\mathbf{u} \in L^2(0, T; \mathbf{V})$ we can improve the previous estimate. Indeed, we have

$$\begin{aligned} \langle \nabla \cdot (\phi \mathbf{u}), w \rangle &= \langle \phi \mathbf{u}, \nabla w \rangle \leq \|\nabla w\| \|\mathbf{u}\|_{L^3} \|\phi\|_{L^6} \\ &\leq \mathcal{Q}(R) \|\nabla w\| \|\mathbf{u}\|^{1/2} \|\mathbf{u}\|_1^{1/2}, \end{aligned}$$

providing

$$\begin{aligned} \left| \int_0^T \langle \nabla \cdot (\phi \mathbf{u}), w \rangle dt \right| &\leq \mathcal{Q}(R) \int_0^T \|\nabla w\| \|\mathbf{u}\|^{1/2} \|\mathbf{u}\|_1^{1/2} dt \\ &\leq \mathcal{Q}(R) \left(\int_0^T \|\nabla w\|^2 dt \right)^{1/2} \left(\int_0^T \|\mathbf{u}\| \|\mathbf{u}\|_1 dt \right)^{1/2} \\ &\leq \frac{C_T}{\nu^{1/4}} \left(\int_0^T \|\nabla w\|^2 dt \right)^{1/2}. \end{aligned}$$

Therefore, if $\nu > 0$, then

$$\nabla \cdot (\phi \mathbf{u}) \in L^2(0, T; H^{-1}).$$

The term $\phi \nabla \mu$. Let $\mathbf{v} \in \mathbf{H}$. Thanks to Agmon's inequality, we infer

$$\langle \phi \nabla \mu, \mathbf{v} \rangle \leq \|\mathbf{v}\| \|\nabla \mu\| \|\phi\|_{L^\infty} \leq \|\mathbf{v}\| \|\nabla \mu\| \|\phi\|_1^{1/2} \|\phi\|_2^{1/2} \leq \mathcal{Q}(R) \|\mathbf{v}\| \|\nabla \mu\| \|\phi\|_2^{1/2}.$$

On account of (3.5), we can estimate as follows

$$\begin{aligned}
\left| \int_0^T \langle \phi \nabla \mu, \mathbf{v} \rangle dt \right| &\leq \mathcal{Q}(R) \int_0^T \|\mathbf{v}\| \|\nabla \mu\| \|\phi\|_2^{1/2} dt \\
&\leq \mathcal{Q}(R) \left(\int_0^T \|\mathbf{v}\|^{8/3} dt \right)^{3/8} \left(\int_0^T \|\nabla \mu\|^2 dt \right)^{1/2} \left(\int_0^T \|\phi\|_2^4 dt \right)^{1/8} \\
&\leq \mathcal{Q}_T(R) \left(\int_0^T \|\mathbf{v}\|^{8/3} dt \right)^{3/8},
\end{aligned}$$

which yields, independently of ν ,

$$\phi \nabla \mu \in L^{8/5}(0, T; \mathbf{H}).$$

4. WELL-POSEDNESS FOR $\nu > 0$

Aim of this section is proving Theorem 2.5, relying on the formal a priori estimates in the previous section. This is done in two steps: first existence of a (global) solution and then its uniqueness are proven.

4.1. Existence. When $\nu > 0$, a weak solution to (1.1)-(1.7) can be constructed through a standard Galerkin procedure. More precisely, let \mathbf{H}_m be the m -dimensional subspace of \mathbf{V} generated by the first m eigenfunctions of the strictly positive operator \mathfrak{A} defined by

$$\mathfrak{A}\mathbf{u} = -\nu \mathfrak{P} \Delta \mathbf{u} + \mathbf{u}, \quad \text{dom}(\mathfrak{A}) = \mathbf{W},$$

where \mathfrak{P} is the orthogonal projector on the space of divergence-free functions

$$\mathfrak{P}: (\mathbf{H})^3 \rightarrow \mathbf{H},$$

and \mathbf{W} is the closure of \mathcal{V} in the topology of $(\mathbf{H}^2)^3$. Analogously, let H_m be the subspace of \mathbf{H} generated by the first m eigenfunction of the Dirichlet operator with homogeneous Neumann boundary conditions. We denote by (ϕ_m, \mathbf{u}_m) , $m \in \mathbb{N}$, the solution of the approximating system

$$\begin{aligned}
\langle \partial_t \phi_m, w \rangle + \langle \nabla \cdot (\phi_m \mathbf{u}_m), w \rangle + \langle \nabla \mu_m, \nabla w \rangle &= 0, \quad \forall w \in H_m, \\
\mu_m &= -\Delta \phi_m + P_m f(\phi_m), \\
\nu \langle \nabla \mathbf{u}_m, \nabla \varphi \rangle + \eta \langle \mathbf{u}_m, \varphi \rangle &= -\langle \phi_m \nabla \mu_m, \varphi \rangle, \quad \forall \varphi \in \mathbf{H}_m,
\end{aligned}$$

with initial conditions

$$\phi_m(0) = P_m \phi_0,$$

where $P_m : \mathbf{H} \rightarrow H_m$ denotes the orthogonal projection on H_m . This system of ODEs has a unique (local) solution. Since the a priori estimates of the previous section also hold for the solutions of the finite-dimensional approximation, we deduce that the solution is indeed defined on $[0, T]$ for every $T > 0$, and the following (uniform w.r.t. m) estimates

hold

$$\begin{aligned}\phi_m &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^3), \\ \partial_t \phi_m &\in L^2(0, T; H^{-1}), \\ \mathbf{u}_m &\in L^2(0, T; \mathbf{V}).\end{aligned}$$

Such estimates are enough to pass to the limit in the Galerkin scheme by standard compactness theorems. We refer the reader to Section 7 where the required argument is detailed in a weaker setting.

As a consequence we obtain the existence of a global weak solution (ϕ, \mathbf{u}) . Furthermore, the well-known Aubin-Lions lemma gives $\phi \in \mathcal{C}([0, T], H^1)$.

4.2. Continuous Dependence and Uniqueness. Let $\nu > 0$ and $\eta > 0$ be fixed, and consider (ϕ_1, \mathbf{u}_1) and (ϕ_2, \mathbf{u}_2) two weak solutions to the CHB system such that $\langle \phi_1(0) \rangle = \langle \phi_2(0) \rangle$. Their difference $\bar{\phi} = \phi_1 - \phi_2$, $\bar{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$ solves a.e. $t \in [0, T]$

$$(4.1) \quad \begin{aligned} \langle \partial_t \bar{\phi}(t), w \rangle + \langle \nabla \cdot (\phi_1(t) \bar{\mathbf{u}}(t)), w \rangle + \langle \nabla \cdot (\bar{\phi}(t) \mathbf{u}_2(t)), w \rangle \\ + \langle \nabla \bar{\mu}(t), \nabla w \rangle = 0, \quad \forall w \in H^1, \end{aligned}$$

$$(4.2) \quad \begin{aligned} \nu \langle \nabla \bar{\mathbf{u}}(t), \nabla \mathbf{v} \rangle + \eta \langle \bar{\mathbf{u}}(t), \mathbf{v} \rangle \\ = -\langle \phi_1(t) \nabla \bar{\mu}(t), \mathbf{v} \rangle - \langle \bar{\phi}(t) \nabla \mu_2(t), \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}, \end{aligned}$$

where

$$\bar{\mu} = -\Delta \bar{\phi} + [f(\phi_1) - f(\phi_2)].$$

and $\langle \bar{\phi} \rangle = 0$.

Taking $w = -\Delta \bar{\phi}$ in (4.1), we get

$$\frac{d}{dt} \frac{1}{2} \|\nabla \bar{\phi}\|^2 + \langle \phi_1 \bar{\mathbf{u}}, \nabla \Delta \bar{\phi} \rangle + \langle \bar{\phi} \mathbf{u}_2, \nabla \Delta \bar{\phi} \rangle - \langle \nabla \bar{\mu}, \nabla \Delta \bar{\phi} \rangle = 0,$$

with

$$\langle \nabla \bar{\mu}, \nabla \Delta \bar{\phi} \rangle = -\|\nabla \Delta \bar{\phi}\|^2 + \langle \nabla [f(\phi_1) - f(\phi_2)], \nabla \Delta \bar{\phi} \rangle.$$

Thus we obtain

$$(4.3) \quad \frac{d}{dt} \frac{1}{2} \|\nabla \bar{\phi}\|^2 + \|\nabla \Delta \bar{\phi}\|^2 = -\langle \phi_1 \bar{\mathbf{u}}, \nabla \Delta \bar{\phi} \rangle - \langle \bar{\phi} \mathbf{u}_2, \nabla \Delta \bar{\phi} \rangle + \langle \nabla [f(\phi_1) - f(\phi_2)], \nabla \Delta \bar{\phi} \rangle.$$

Taking $\mathbf{v} = \bar{\mathbf{u}}$ in (4.2) yields

$$\nu \|\nabla \bar{\mathbf{u}}\|^2 + \eta \|\bar{\mathbf{u}}\|^2 = -\langle \phi_1 \nabla \bar{\mu}, \bar{\mathbf{u}} \rangle - \langle \bar{\phi} \nabla \mu_2, \bar{\mathbf{u}} \rangle.$$

Note that, by definition of $\bar{\mu}$, we have

$$-\langle \phi_1 \nabla \bar{\mu}, \bar{\mathbf{u}} \rangle = \langle \phi_1 \nabla \Delta \bar{\phi}, \bar{\mathbf{u}} \rangle - \langle \phi_1 \nabla [f(\phi_1) - f(\phi_2)], \bar{\mathbf{u}} \rangle,$$

so that the terms $\pm \langle \phi_1 \bar{\mathbf{u}}, \nabla \Delta \bar{\phi} \rangle$ get canceled when adding with (4.3). Therefore we end up with

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\nabla \bar{\phi}\|^2 + \|\nabla \Delta \bar{\phi}\|^2 + \nu \|\nabla \bar{\mathbf{u}}\|^2 + \eta \|\bar{\mathbf{u}}\|^2 \\ = -\langle \bar{\phi} \mathbf{u}_2, \nabla \Delta \bar{\phi} \rangle - \langle \bar{\phi} \nabla \mu_2, \bar{\mathbf{u}} \rangle - \langle \phi_1 \nabla [f(\phi_1) - f(\phi_2)], \bar{\mathbf{u}} \rangle + \langle \nabla [f(\phi_1) - f(\phi_2)], \nabla \Delta \bar{\phi} \rangle. \end{aligned}$$

We now estimate the right hand side in light of the energy estimates in Section 3. This, in particular, gives

$$\sup_{t \geq 0} (\|\phi_1(t)\|_1 + \|\phi_2(t)\|_1) \leq \mathcal{Q}(R),$$

with $R = \max\{\|\phi_1(0)\|_1, \|\phi_2(0)\|_1\}$. First of all, we have

$$-\langle \bar{\phi} \mathbf{u}_2, \nabla \Delta \bar{\phi} \rangle \leq \|\bar{\phi}\|_1 \|\mathbf{u}_2\|^{1/2} \|\mathbf{u}_2\|_1^{1/2} \|\nabla \Delta \bar{\phi}\| \leq \frac{1}{4} \|\nabla \Delta \bar{\phi}\|^2 + \frac{h(t)}{\nu^{1/2}} \|\bar{\phi}\|_1^2,$$

where $h(t) := c\nu^{1/2} \|\mathbf{u}_2(t)\| \|\mathbf{u}_2(t)\|_1$ and $c > 0$ is independent of ν .

Next, observe that the following estimate holds

$$(4.4) \quad -\langle \bar{\phi} \nabla \mu_2, \bar{\mathbf{u}} \rangle \leq \|\bar{\phi}\|_1 \|\bar{\mathbf{u}}\|_{L^3} \|\nabla \mu_2\| \leq \frac{\nu}{2} \|\nabla \bar{\mathbf{u}}\|^2 + \frac{\eta}{4} \|\bar{\mathbf{u}}\|^2 + \frac{k(t)}{\eta^{1/2} \nu^{1/2}} \|\bar{\phi}\|_1^2,$$

where $k(t) := c \|\nabla \mu_2\|^2$ for some $c > 0$, independent of ν .

In order to deal with the term

$$\begin{aligned} \langle \nabla[f(\phi_1) - f(\phi_2)], \nabla \Delta \bar{\phi} \rangle &\leq \|\nabla[f(\phi_1) - f(\phi_2)]\| \|\nabla \Delta \bar{\phi}\| \\ &\leq \frac{1}{4} \|\nabla \Delta \bar{\phi}\|^2 + C \|\nabla[f(\phi_1) - f(\phi_2)]\|^2, \end{aligned}$$

we observe that

$$\|\nabla[f(\phi_1) - f(\phi_2)]\|^2 \leq \|[f'(\phi_1) - f'(\phi_2)] \nabla \phi_1\|^2 + \|f'(\phi_2) \nabla \bar{\phi}\|^2.$$

We estimate the latter term on the right hand side in light of (2.1), (3.1) and interpolation, that is,

$$\begin{aligned} \|f'(\phi_2) \nabla \bar{\phi}\|^2 &\leq c \int_{\Omega} (1 + |\phi_2|^4) |\nabla \bar{\phi}|^2 \\ &\leq c(1 + \|\phi_2\|_{L^6}^4) \|\nabla \bar{\phi}\|_{L^6}^2 \\ &\leq \mathcal{Q}(R) \|\bar{\phi}\|_1 \|\bar{\phi}\|_3 \\ &\leq \frac{1}{4} \|\nabla \Delta \bar{\phi}\|^2 + \mathcal{Q}(R) \|\bar{\phi}\|_1^2, \end{aligned}$$

and arguing analogously for the former, we get

$$\begin{aligned} \|[f'(\phi_1) - f'(\phi_2)] \nabla \phi_1\|^2 &\leq c \int_{\Omega} |(1 + |\phi_1| + |\phi_2|) \bar{\phi} \nabla \phi_1|^2 \\ &\leq \mathcal{Q}(R) \left(\int_{\Omega} |\bar{\phi}|^6 \right)^{1/3} \left(\int_{\Omega} |\nabla \phi_1|^6 \right)^{1/3} \\ &\leq \mathcal{Q}(R) \|\bar{\phi}\|_1^2 \|\phi_1\|_2^2. \end{aligned}$$

This proves

$$(4.5) \quad \|\nabla[f(\phi_1) - f(\phi_2)]\|^2 \leq \ell(t) \|\bar{\phi}\|_1^2 + \frac{1}{4} \|\nabla \Delta \bar{\phi}\|^2,$$

where $\ell(t) := \mathcal{Q}(R)(1 + \|\phi_1(t)\|_2^2)$. In order to control the remaining term, we exploit (4.5) in the following way

$$\begin{aligned}
 (4.6) \quad -\langle \phi_1 \nabla[f(\phi_1) - f(\phi_2)], \bar{\mathbf{u}} \rangle &\leq \|\phi_1\|_{L^6} \|\nabla[f(\phi_1) - f(\phi_2)]\| \|\bar{\mathbf{u}}\|_{L^3} \\
 &\leq \frac{\nu^{1/2}}{2} \|\bar{\mathbf{u}}\| \|\nabla \bar{\mathbf{u}}\| + \frac{\mathcal{Q}(R)}{\nu^{1/2}} \|\nabla[f(\phi_1) - f(\phi_2)]\|^2 \\
 &\leq \frac{\nu}{2} \|\nabla \bar{\mathbf{u}}\|^2 + \frac{\eta}{2} \|\bar{\mathbf{u}}\|^2 + \frac{\ell(t)}{\nu^{1/2} \eta^{1/2}} \|\bar{\phi}\|_1^2.
 \end{aligned}$$

Collecting the above estimates we get

$$(4.7) \quad \frac{d}{dt} \|\bar{\phi}\|_1^2 + \frac{\nu}{2} \|\nabla \bar{\mathbf{u}}\|^2 + \frac{\eta}{2} \|\bar{\mathbf{u}}\|^2 \leq \frac{g(t)}{\nu^{1/2} \eta^{1/2}} \|\bar{\phi}\|_1^2,$$

where $g(t) := h(t) + k(t) + \ell(t)$, on account of (3.1) and (3.2), satisfies

$$\int_0^T g(y) dy \leq \mathcal{Q}_T(R).$$

Hence an application of the standard Gronwall lemma gives

$$\|\phi_1(t) - \phi_2(t)\|_1^2 \leq \|\phi_1(0) - \phi_2(0)\|_1^2 e^{\nu^{-1/2} \eta^{-1/2} \int_0^t g(y) dy},$$

which proves (2.10). An integration of (4.7) yields the further bound (2.11). Finally, letting $\phi_1(0) = \phi_2(0)$ in (2.10) and (2.11) we obtain $\phi_1(t) = \phi_2(t)$ and $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ for almost every t , i.e., uniqueness.

We observe that, when $\eta = 0$ (see Remark 2.7), the only changes needed in the proof of the continuous dependence estimate are in (4.4) and in (4.6), which now become

$$\begin{aligned}
 -\langle \bar{\phi} \nabla \mu_2, \bar{\mathbf{u}}, \bar{\mathbf{u}} \rangle &\leq \frac{\nu}{4} \|\nabla \bar{\mathbf{u}}\|^2 + \frac{k(t)}{\nu} \|\bar{\phi}\|_1^2 \\
 -\langle \phi_1 \nabla[f(\phi_1) - f(\phi_2)], \bar{\mathbf{u}} \rangle &\leq \frac{\nu}{2} \|\nabla \bar{\mathbf{u}}\|^2 + \frac{\ell(t)}{\nu} \|\bar{\phi}\|_1^2.
 \end{aligned}$$

4.3. The semigroup $S_\nu(t)$. Let $I \in \mathbb{R}$ and consider the subspace of H^1

$$V_I = \{\phi \in H^1 : \langle \phi \rangle = I\}.$$

An immediate consequence of the results of Section 4.2 is that, for any fixed $\nu > 0$, system (1.1)-(1.6) generates a semigroup

$$S_\nu(t) : V_I \rightarrow V_I$$

defined by

$$S_\nu(t)\phi_0 = \phi(t),$$

where (ϕ, \mathbf{u}) is the unique global (weak) solution to system (1.1)-(1.6). Furthermore, owing to the continuous dependence estimate (2.10), the semigroup is strongly continuous, namely,

$$S_\nu(t) \in \mathcal{C}(V_I, V_I).$$

Besides, the energy estimates of Section 3 yield in particular

$$(4.8) \quad \|S_\nu(t)\phi_0\|_1 = \|\phi(t)\|_1 \leq c, \quad \forall t \geq 0,$$

where, from now on, $c \geq 0$ denotes a generic constant that may depend on $\|\phi_0\|_1$ but is independent of the particular ϕ_0 .

4.4. Absorbing sets for $S_\nu(t)$. If the nonlinearity f satisfies further dissipativity assumptions stronger than (2.2), it is possible to prove that the energy estimate (4.8) is in fact *independent* of $\|\phi_0\|_1$ when t is large enough. More precisely, under conditions (4.10)-(4.11) below, we prove that

$$(4.9) \quad \|S_\nu(t)\phi_0\|_1 \leq \mathcal{Q}(\|\phi_0\|_1)e^{-kt/2} + R_I, \quad \forall t \geq 0,$$

for some $k > 0$, where $R_I > 0$ depends on I but is independent of ϕ_0 . This can be subsumed by saying that the ball

$$\mathcal{B} = \{\phi \in V_I : \|\phi\|_1 \leq R_I + 1\}$$

is a (bounded) absorbing set for the semigroup $S_\nu(t)$ acting on V_I , namely, for every $R > 0$ there exists $t_R > 0$ such that

$$S_\nu(t)\phi_0 \in \mathcal{B}, \quad \forall t \geq t_R,$$

for every $\phi_0 \in V_I$ with $\|\phi_0\|_1 \leq R$.

Let us formulate the additional hypotheses on f by supposing that for some $c_0 \geq 0$, $c_i > 0$, $i = 1, 2$ and $q > 2$ there hold

$$(4.10) \quad f(s)s \geq c_1 F(s) - c_0,$$

and

$$(4.11) \quad F(s) \geq c_2 |s|^q - c_0,$$

for all $s \in \mathbb{R}$. Notice that the usual double well potential (2.6) satisfies these requirements. Set now

$$E(z) = \frac{1}{2} \|\nabla z\|^2 + \langle F(z), 1 \rangle$$

for $z \in H^1$. Then we have

Proposition 4.1. *Let the assumptions of Theorem 2.5 hold and assume that f satisfies, in addition, (4.10)–(4.11). Let (ϕ, \mathbf{u}) be the weak solution to the CHB system originating from $\phi_0 \in H^1$, with $I := \langle \phi_0 \rangle$. Then, the energy E satisfies the following dissipative estimate*

$$(4.12) \quad E(\phi(t)) \leq E(\phi_0)e^{-kt} + K_I, \quad \forall t \geq 0,$$

where $k > 0$ is independent of the initial data, and $K_I > 0$ depends on I but is independent of ϕ_0 .

Proof. Along the proof $c \geq 0$ denotes a generic constant independent of ϕ_0 . By multiplying the equation (1.2) for μ by ϕ in H we obtain

$$\langle \mu, \phi \rangle = \|\nabla \phi\|^2 + \langle f(\phi), \phi \rangle.$$

In light of (4.10) and (4.11), we get

$$\begin{aligned} \langle \mu, \phi \rangle &\geq \|\nabla \phi\|^2 + c_1 \langle F(\phi), 1 \rangle - c \\ &\geq \|\nabla \phi\|^2 + \frac{c_1}{2} \langle F(\phi), 1 \rangle + \frac{c_1 c_2}{2} \|\phi\|_{L^q}^q - c. \end{aligned}$$

We now assume $I = 0$. Then, we have

$$\langle \mu, \phi \rangle = \langle \mu - \langle \mu \rangle, \phi \rangle \leq c \|\nabla \mu\| \|\phi\| \leq \|\nabla \mu\|^2 + c \|\phi\|^2.$$

Collecting the two inequalities we obtain

$$\|\nabla \phi\|^2 + \frac{c_1}{2} \langle F(\phi), 1 \rangle \leq \|\nabla \mu\|^2 + c \|\phi\|^2 - \frac{c_1 c_2}{2} \|\phi\|_{L^q}^q + c.$$

Notice that, since $q > 2$, the inequality

$$c \|\phi\|^2 - \frac{c_1 c_2}{2} \|\phi\|_{L^q}^q \leq c,$$

holds. Thus we end up with

$$(4.13) \quad \|\nabla \phi\|^2 + \frac{c_1}{2} \langle F(\phi), 1 \rangle \leq \|\nabla \mu\|^2 + c.$$

Now, recalling (3.3), we know that

$$\frac{d}{dt} E(\phi) + \|\nabla \mu\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \eta \|\mathbf{u}\|^2 = 0,$$

and adding this energy identity to (4.13) we obtain the differential inequality

$$\frac{d}{dt} E(\phi) + k E(\phi) \leq c,$$

for some $k > 0$. Hence the Gronwall lemma implies

$$(4.14) \quad E(\phi(t)) \leq E(\phi_0) e^{-kt} + c, \quad \forall t \geq 0,$$

proving (4.12) in the case $I = 0$. If $I \neq 0$, let $\tilde{\phi} = \phi - I$ and observe that the pair $(\tilde{\phi}, \mathbf{u})$ is a weak solution to the same problem, but with potential

$$\tilde{F}(s) := F(s + I) - F(I),$$

and initial data $\tilde{\phi}_0 := \phi_0 - I$, satisfying $\langle \tilde{\phi}_0 \rangle = 0$. We can thus exploit (4.14) for $\tilde{\phi}$, so obtaining

$$\tilde{E}(\tilde{\phi}(t)) \leq \tilde{E}(\phi_0) e^{-kt} + c, \quad \forall t \geq 0,$$

where

$$\tilde{E}(z) := \frac{1}{2} \|\nabla z\|^2 + \langle \tilde{F}(z), 1 \rangle,$$

proving (4.12). □

By relying on Proposition 4.1, on account of (2.1) and (2.2), we easily get (4.9). Therefore we can say that the dynamical system $(V_I, S_\nu(t))$ is dissipative for any fixed $I \in \mathbb{R}$ and $\nu > 0$.

5. HIGHER ORDER ESTIMATES

Here we proceed formally relying on the Galerkin approximation scheme introduced in the previous section.

For the sake of simplicity, from now on we set $\eta = 1$ (see Remark 6.7, however).

Proposition 5.1. *Let the assumptions of Theorem 2.5 hold and suppose, in addition, $f \in \mathcal{C}^2(\mathbb{R})$ satisfying (2.4). Then the following estimate holds*

$$(5.1) \quad \|\phi(t)\|_2 + \int_t^{t+1} \|\phi(y)\|_4^2 dy \leq c \left(1 + \frac{1}{\nu}\right), \quad \forall t \geq 1.$$

Proof. Taking $\mathbf{v} = \mathbf{u}$ in equation (2.8) we get

$$(5.2) \quad \nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2 = \langle \mu \nabla \phi, \mathbf{u} \rangle.$$

By (2.1) and (4.8) we have

$$\|\mu\| \leq \|\Delta \phi\| + \|f(\phi)\| \leq c(1 + \|\Delta \phi\|).$$

Thus, for $\nu > 0$, we can estimate the latter term as follows

$$\begin{aligned} \langle \mu \nabla \phi, \mathbf{u} \rangle &\leq \|\mu\| \|\nabla \phi\|_{L^6} \|\mathbf{u}\|_{L^3} \\ &\leq c(1 + \|\Delta \phi\|) \|\Delta \phi\| \|\mathbf{u}\|^{1/2} \|D\mathbf{u}\|^{1/2} \\ &\leq \frac{1}{2} \|\mathbf{u}\|^2 + \frac{\nu}{2} \|\nabla \mathbf{u}\|^2 + \frac{c}{\nu^{1/2}} (1 + \|\Delta \phi\|^2) \|\Delta \phi\|^2. \end{aligned}$$

From this we deduce

$$(5.3) \quad \nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2 \leq \frac{c}{\nu^{1/2}} (1 + \|\Delta \phi\|^2) \|\Delta \phi\|^2.$$

Let us now take $w = \Delta^2 \phi$ in equation (2.7). This yields

$$\frac{1}{2} \frac{d}{dt} \|\Delta \phi\|^2 + \|\Delta^2 \phi\|^2 = \langle \Delta f(\phi), \Delta^2 \phi \rangle + \langle \mathbf{u} \nabla \phi, \Delta^2 \phi \rangle.$$

On the other hand, we have

$$\langle \Delta f(\phi), \Delta^2 \phi \rangle \leq \frac{1}{4} \|\Delta^2 \phi\|^2 + c \|\Delta f(u)\|^2,$$

where $\|\Delta f(\phi)\|^2$ can be controlled in the following way. Observe that

$$\Delta f(\phi) = \nabla(f'(\phi) \nabla \phi) = f''(\phi) |\nabla \phi|^2 + f'(\phi) \Delta \phi.$$

Then, using (2.4), by the Agmon inequality we get

$$\begin{aligned} \|f''(\phi) |\nabla \phi|^2\| &\leq c(1 + \|\phi\|_{L^\infty}) \|\nabla \phi\|_{L^4}^2 \leq c(1 + \|\Delta \phi\|^{1/2}) \|\Delta \phi\|^{3/2}, \\ \|f'(\phi) \Delta \phi\| &\leq c(1 + \|\phi\|_{L^\infty}^2) \|\Delta \phi\| \leq c(1 + \|\Delta \phi\|) \|\Delta \phi\|. \end{aligned}$$

Therefore, we obtain

$$(5.4) \quad \|\Delta f(\phi)\|^2 \leq c(1 + \|\Delta \phi\|^2) \|\Delta \phi\|^3.$$

In order to deal with the remaining term, exploiting (5.3) we find

$$\begin{aligned}
\langle \mathbf{u} \nabla \phi, \Delta^2 \phi \rangle &\leq c \|\mathbf{u}\|_{L^3} \|\nabla \phi\|_{L^6} \|\Delta^2 \phi\| \\
&\leq \|\mathbf{u}\|^{1/2} \|D\mathbf{u}\|^{1/2} \|\Delta \phi\| \|\Delta^2 \phi\| \\
&\leq \frac{c}{\nu^{1/2}} (1 + \|\Delta \phi\|) \|\Delta \phi\|^2 \|\Delta^2 \phi\| \\
&\leq \frac{1}{4} \|\Delta^2 \phi\|^2 + \frac{c}{\nu} (1 + \|\Delta \phi\|^2) \|\Delta \phi\|^4.
\end{aligned}$$

We thus end up with the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\Delta \phi\|^2 + \frac{1}{2} \|\Delta^2 \phi\|^2 \leq c(1 + \|\Delta \phi\|^2) \|\Delta \phi\|^3 + \frac{c}{\nu} (1 + \|\Delta \phi\|^2) \|\Delta \phi\|^4.$$

Recalling that $\phi \in L^4(0, T; H^2)$ (see (3.5)), Lemma 2.13 yields the claimed result. \square

Remark 5.2. Estimate (5.1) entails that the weak solution ϕ is indeed a strong one for $t \geq 1$ (see (1.1)). In addition, if Ω is of class $C^{1,1}$, then the regularity of $\mu \nabla \phi$ implies that the weak solution \mathbf{u} to (2.8) belongs to $L_{loc}^2((1, \infty); (H^2(\Omega))^3)$ and the pressure p (see Remark 2.4), unique up to a constant, belongs to $L_{loc}^2((1, \infty); H^1(\Omega))$ (see, e.g., [5, Theorem IV.5.8]). Thus equation (1.3) is also satisfied almost everywhere if $t \geq 1$.

We conclude this section by proving the existence of the global attractor, namely,

Theorem 5.3. *Let f satisfy all the assumptions in Proposition 4.1 and Proposition 5.1. Then the dynamical system $(V_I, S_\nu(t))$ possesses a (unique) global attractor \mathcal{A} which is bounded in H^2 .*

Proof. On account of the assumptions on f , thanks to Proposition 4.1 and Proposition 5.1, we infer the existence of a *compact* absorbing set (bounded in H^2) for the semigroup $S_\nu(t)$. Hence, by standard results (see, e.g., [25]) the proof follows. \square

Remark 5.4. We recall that the global attractor \mathcal{A} is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S_\nu(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity, namely,

$$\forall B \subset V_I \text{ bounded, } \quad \text{dist}(S_\nu(t)B, \mathcal{A}) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where *dist* denotes the Hausdorff semi-distance between sets in H^1 .

6. CONVERGENCE TO EQUILIBRIA

Along this section we let $\nu > 0$ be fixed omitting in the notation the dependence on ν . Let $\phi_0 \in H^1$ and let $\phi(\cdot) = S(\cdot)\phi_0$ be the global weak solution to the CHB system originating from ϕ_0 . The ω -limit set of ϕ_0 is defined as

$$\omega(\phi_0) = \{\phi^* \in H^1 : \phi(t_n) \rightarrow \phi^* \text{ in } H^1, \text{ for some } \{t_n\}_{n \in \mathbb{N}}, t_n \rightarrow \infty\}.$$

The set of stationary points associated with ϕ_0 is

$$\mathcal{S}(\phi_0) = \{z \in H^2 : -\Delta z + f(z) = \text{const}, \langle z \rangle = \langle \phi_0 \rangle\}.$$

Aim of this section is to prove Theorem 2.8, showing in particular that each ω -limit set consists of one single stationary point.

The proof consists of several steps.

Lyapunov functional. We first prove the existence of a *Lyapunov functional* for the semigroup $S(t)$. This is a function $L \in \mathcal{C}(H^1, \mathbb{R})$ satisfying the following properties:

- (i) $L(S(t)z) \leq L(z)$, for every $z \in H^1$,
- (ii) $L(S(t)z) = L(z)$ for every $t \geq 0$ implies that z is a stationary point for $S(t)$ (namely $S(t)z = z$ for every $t \geq 0$).

Proposition 6.1. *The functional*

$$E(z) = \frac{1}{2} \|\nabla z\|^2 + \langle F(z), 1 \rangle$$

with $z \in H^1$ is a *Lyapunov functional* for $S(t)$. Besides, if $E(S(t)z) = E(z)$ for every $t \geq 0$, then $z \in \mathcal{S}$ where

$$\mathcal{S} = \{z \in H^2 : -\Delta z + f(z) = \text{const} = \langle f(z) \rangle\}.$$

Proof. Let $\phi(t) = S(t)z$ for $t \geq 0$. Recalling (3.3) we have

$$(6.1) \quad \frac{d}{dt} E(\phi) + \|\nabla \mu\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2 = 0,$$

hence $E(\phi(\cdot))$ is nonincreasing and (i) follows. In order to show (ii), we assume $E(S(t)z) = E(z)$ for all $t \geq 0$. On account of (6.1) this implies $\mathbf{u}(t) = 0$ and $\nabla \mu(t) = 0$ for almost every $t > 0$. Besides, since $\partial_n \mu = 0$, μ is a constant for almost every $t > 0$, and equation (2.7) ensures that $\partial_t \phi = 0$. Hence $S(t)z = \phi(t)$ is constant for almost every $t > 0$. Owing to the continuity $\phi \in \mathcal{C}([0, \infty), H^1)$ we get $S(t)z = z$ for every $t \geq 0$, proving that z is a stationary point for $S(t)$. The weak continuity $\phi \in \mathcal{C}_w(t_0, \infty; H^2)$ (for any arbitrary $t_0 > 0$) ensures that $z \in H^2$ and that $\mu(t)$ is constant for all $t > 0$. This gives

$$-\Delta z + f(z) = k,$$

for some constant $k \in \mathbb{R}$, and an integration on Ω shows that $k = \langle f(z) \rangle$, concluding the proof. \square

We now prove the following

Proposition 6.2. *For every $\phi_0 \in H^1$, the ω -limit set of ϕ_0 is a nonempty compact subset of H^1 . Besides, if (2.2) holds then*

$$\omega(\phi_0) \subset \mathcal{S}(\phi_0).$$

Proof. On account of Section 5 and thanks to the compact embedding $H^2 \hookrightarrow H^1$, we have that $\omega(\phi_0)$ is compact and there exist $\phi^* \in H^1$ and $t_n \rightarrow \infty$ such that

$$\|\phi(t_n) - \phi^*\|_1 \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where $\phi(t_n) = S(t_n)\phi_0$. The inclusion $\omega(\phi_0) \subset \mathcal{S}(\phi_0)$ follows by the standard theory of gradient systems (see, e.g., [8, Chapter 9]), but we report a short proof for the reader's convenience.

We first notice that $S(t)\omega(\phi_0) \subset \omega(\phi_0)$. Indeed, if $z \in \omega(\phi_0)$ then $S(y_n)\phi_0 \rightarrow z$ for some $y_n \rightarrow \infty$. Since $S(t)$ is continuous for every $t > 0$,

$$S(t + y_n)\phi_0 = S(t)S(y_n)\phi_0 \rightarrow S(t)z,$$

thus $S(t)z \in \omega(\phi_0)$.

Let us consider now the Lyapunov functional E as in Proposition 6.1. Observing that E is bounded below due to the dissipativity condition (2.2) on F , we have

$$(6.2) \quad \lim_{t \rightarrow \infty} E(S(t)\phi_0) = \lim_{n \rightarrow \infty} E(S(t_n)\phi_0) = E(\phi^*) =: E^*,$$

so that

$$E(z) = E^*, \quad \forall z \in \omega(\phi_0).$$

By the forward invariance of $\omega(\phi_0)$ we then obtain $E(S(t)z) = E^*$ for every $t \geq 0$, which means

$$z \in \omega(\phi_0) \quad \Rightarrow \quad E(S(t)z) = E(z), \quad \forall t \geq 0.$$

An application of Proposition 6.1 shows that $z \in \mathcal{S}$. Thus we are only left to prove that $\langle z \rangle = \langle \phi_0 \rangle$. But this is an immediate consequence of the mass conservation (2.9). \square

The key tool in order to guarantee the convergence of trajectories to single stationary states is the following version of the Łojasiewicz-Simon inequality.

Theorem 6.3. *Let (ϕ, \mathbf{u}) be a solution of system (1.1)-(1.4) with initial datum $\phi_0 \in H^1$ and let $z \in \omega(\phi_0) \subset \mathcal{S}(\phi_0)$. If f is real analytic and satisfies (2.5), then there exist $\theta = \theta(z) > 0$, $\theta \in (0, \frac{1}{2})$ and $\varsigma = \varsigma(z) > 0$ such that*

$$(6.3) \quad |E(\phi) - E(z)|^{1-\theta} \leq \|\mathbf{P}(\Delta\phi - f(\phi))\|_{H^{-1}},$$

whenever ϕ fulfills $\|\phi - z\|_1 < \varsigma$. Here $\mathbf{P} : H \rightarrow H$ is defined by $\mathbf{P}(u) = u - \langle u \rangle$.

The reader is referred to [1, Proposition 6.3] for the proof, where a singular potential appears, providing uniform L^∞ bounds for the solutions of the system under study. Although in our setting the potential f is regular, the uniform estimate $\phi \in L^\infty(1, \infty; H^2)$ obtained in Section 5 entails $\phi \in L^\infty(\Omega \times [1, \infty))$, which is sufficient for the above theorem to hold.

6.1. Proof of Theorem 2.8. Let us first observe that from (5.1) we have

$$(6.4) \quad \|\phi(t)\|_2 \leq c, \quad t \geq 1.$$

Besides, in light of Proposition 6.2 and (2.1), we deduce

$$\sup_{z \in \omega(\phi_0)} \|z\|_2 \leq c.$$

We start by proving the first part of the result, namely,

Proposition 6.4. *There exists $\phi^* \in \mathcal{S}(\phi_0)$ such that $\omega(\phi_0) = \{\phi^*\}$.*

Proof. We use a well-known contradiction argument due to [14] (see also [30] and references therein). According to Proposition 6.2, there exists $\phi^* \in H^1$ and $t_n \rightarrow \infty$ such that

$$(6.5) \quad \|\phi(t_n) - \phi^*\|_1 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Besides, by (6.2), we have

$$(6.6) \quad E(\phi(t_n)) \rightarrow E(\phi^*).$$

We will get the proof by showing that the whole trajectory $\phi(\cdot)$ converges to ϕ^* , namely

$$(6.7) \quad \lim_{t \rightarrow \infty} \|\phi(t) - \phi^*\|_1 = 0.$$

To this aim, in light of (6.4), we deduce from equation (2.7) that

$$\|\partial_t \phi\|_{H^{-1}} \leq \|\phi \mathbf{u}\| + \|\nabla \mu\| \leq c(\|\mathbf{u}\| + \|\nabla \mu\|).$$

Moreover, recalling (5.2), we have

$$\nu \|\nabla \mathbf{u}\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \leq \|\phi \nabla \mu\|^2 \leq c \|\nabla \mu\|^2.$$

Combining the last two estimates we find

$$(6.8) \quad \|\partial_t \phi(t)\|_{H^{-1}} \leq c \|\nabla \mu(t)\|, \quad t \geq 1.$$

Suppose that there exists $t_1 \geq 1$ such that $E(\phi(t_1)) = E(\phi^*)$. Since $E(\phi(\cdot))$ is nonincreasing, then

$$E(\phi(t)) = E(\phi^*), \quad \forall t \geq t_1.$$

Hence from (6.1) we get $\|\mathbf{u}\| = \|\nabla \mu\| = 0$ for $t \geq t_1$. On the other hand (6.8) implies $\|\partial_t \phi(t)\|_{H^{-1}} = 0$ for $t \geq t_1$, meaning that $\phi(t)$ is definitively independent of time. In this case, (6.7) directly follows by (6.5).

Otherwise, we assume that

$$(6.9) \quad \Gamma(t) := E(\phi(t)) - E(\phi^*) > 0, \quad \forall t > 1.$$

Thanks to (6.5), for any given $\varsigma > 0$, there is $m = m(\varsigma) \in \mathbb{N}$ such that

$$t_n \geq 1, \quad \text{and} \quad \|\phi(t_n) - \phi^*\|_1 < \varsigma, \quad \forall n \geq m.$$

Defining then, for every $n \geq m$,

$$t_n^* = \sup \{t \geq t_n : \|\phi(\tau) - \phi^*\|_1 < \varsigma, \forall \tau \in [t_n, t]\},$$

we can apply Theorem 6.3 to $\phi(t)$ for every $t \in [t_n, t_n^*]$, so obtaining

$$\begin{aligned} -\frac{1}{\theta} \frac{d}{dt} [\Gamma(t)]^\theta &= -[\Gamma(t)]^{\theta-1} \frac{d}{dt} E(\phi(t)) \\ &= [\Gamma(t)]^{\theta-1} (\|\nabla \mu\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2) \geq \frac{\|\nabla \mu\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2}{\|\mathbf{P}(\Delta \phi - f(\phi))\|_{H^{-1}}} \\ &\geq \frac{\|\nabla \mu\|^2 + \nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2}{c \|\nabla \mu\|} \geq c \|\nabla \mu\|, \end{aligned}$$

which proves

$$(6.10) \quad -\frac{d}{dt} [\Gamma(t)]^\theta = -\theta [\Gamma(t)]^{\theta-1} \frac{d}{dt} \Gamma(t) \geq c \|\nabla \mu\|,$$

for all $t \in [t_n, t_n^*]$. Note that in the above computations we are exploiting the control

$$\|\mathbf{P}(\Delta \phi - f(\phi))\|_{H^{-1}} = \|\mathbf{P}\mu\|_{H^{-1}} \leq c \|\nabla \mu\|.$$

Combining the latter relation with (6.8) and integrating on (t_n, t_n^*) yields

$$\int_{t_n}^{t_n^*} \|\partial_t \phi(y)\|_{H^{-1}} dy \leq c \int_{t_n}^{t_n^*} \|\nabla \mu\| dy \leq [\Gamma(t_n^*)]^\theta - [\Gamma(t_n)]^\theta.$$

Since the last term vanishes as $n \rightarrow \infty$ in light of (6.2) and (6.6), we obtain

$$\lim_{n \rightarrow \infty} \int_{t_n}^{t_n^*} \|\partial_t \phi(y)\|_{H^{-1}} dy = 0.$$

Let us now prove that $t_n^* = \infty$ for every n . If not, we can compute

$$\limsup_{n \rightarrow \infty} \|\phi(t_n^*) - \phi^*\|_{H^{-1}} \leq \lim_{n \rightarrow \infty} \left(\|\phi(t_n) - \phi^*\|_{H^{-1}} + \int_{t_n}^{t_n^*} \|\partial_t \phi(y)\|_{H^{-1}} dy \right) = 0,$$

hence $\phi(t_n^*) \rightarrow \phi^*$ in H^{-1} . Since $\phi(t_n^*)$ is precompact in H^1 , this implies that $\phi(t_n^*) \rightarrow \phi^*$ in H^1 , which contradicts the definition of t_n^* . Thus, $t_n^* = \infty$ for all $n \geq m$ and we end up with

$$\lim_{n \rightarrow \infty} \int_{t_n}^{\infty} \|\partial_t \phi(y)\|_{H^{-1}} dy = 0.$$

We therefore conclude that the trajectory $\phi(t)$ converges to ϕ^* in H^{-1} when $t \rightarrow \infty$. By the compactness of $\{\phi(t)\}_{t \geq 1}$ in H^1 and by the uniqueness of limits, we have $\phi(t) \rightarrow \phi^*$ in H^1 . \square

The rate of convergence of the trajectory to the equilibrium is witnessed by the following:

Proposition 6.5. *Let $\theta = \theta(\phi^*) \in (0, \frac{1}{2})$ as provided by Theorem 6.3. Then,*

$$(6.11) \quad \|\phi(t) - \phi^*\|_1 \leq \frac{c}{(1+t)^{\theta/(1-2\theta)}},$$

for some $c = c(\phi_0) \geq 0$ and every $t \geq t^*$, for some $t^* > 0$.

Proof. We use the same setting as in the proof of Proposition 6.4 (cf. also [30, 5.2]). Assuming $\Gamma > 0$ as in (6.9), we deduce by (6.3) and (6.10) that

$$\frac{d}{dt} \Gamma(t) + c \Gamma^{2(1-\theta)}(t) \leq 0, \quad \forall t \geq t^*,$$

for some $c > 0$ and some $t^* > 0$, which yields

$$\Gamma(t) \leq \frac{c}{(1+t)^{1/(1-2\theta)}}, \quad t \geq t^*.$$

Collecting (6.8) and (6.10), an integration on $[t, \infty)$ provides

$$\|\phi(t) - \phi^*\|_{H^{-1}} \leq \int_t^{\infty} \|\partial_t \phi(y)\|_{H^{-1}} dy \leq c \int_t^{\infty} \|\nabla \mu(y)\| dy \leq c \int_t^{\infty} -\frac{d}{dt} [\Gamma(y)]^\theta dy.$$

Therefore we get

$$(6.12) \quad \|\phi(t) - \phi^*\|_{H^{-1}} + \int_t^{\infty} \|\nabla \mu(y)\| dy \leq \frac{c}{(1+t)^{\theta/(1-2\theta)}}, \quad t \geq t^*.$$

Let us now set

$$\Phi(t) = \phi(t) - \phi^*,$$

and observe that, for $t \geq t^*$ and almost everywhere in Ω (cf. Remark 5.2), there holds

$$(6.13) \quad \partial_t \Phi + \mathbf{u} \cdot \nabla (\Phi + \phi^*) = \Delta (-\Delta \Phi + [f(\phi) - f(\phi^*)]).$$

Recalling (5.2), owing to (6.4), we have

$$\begin{aligned}
\nu \|D\mathbf{u}\|^2 + \|\mathbf{u}\|^2 &= \langle \phi \nabla \mu, \mathbf{u} \rangle \\
&\leq \|\nabla \mu\| \|\mathbf{u}\| \|\phi\|_{L^\infty} \\
&\leq \|\nabla \mu\| \|\mathbf{u}\| \|\phi\|_1^{1/2} \|\phi\|_2^{1/2} \\
&\leq \frac{1}{2} \|\mathbf{u}\|^2 + c \|\nabla \mu\|^2.
\end{aligned}$$

Besides, since $\phi^* \in \mathcal{S}(\phi_0)$, then $\mu^* := -\Delta \phi^* + f(\phi^*)$ is constant, and we can estimate

$$\begin{aligned}
\|\nabla \mu\| &= \|\nabla(\mu - \mu^*)\| \leq \|\nabla \Delta \Phi\| + \|\nabla(f(\phi) - f(\phi^*))\| \\
&\leq \|\nabla \Delta \Phi\| + c \|\nabla \Phi\| \\
&\leq \|\nabla \Delta \Phi\| + c \|\Phi\|_{H^{-1}}^{1/2} \|\nabla \Delta \Phi\|^{1/2} \\
&\leq c \|\nabla \Delta \Phi\| + \|\Phi\|_{H^{-1}}.
\end{aligned}$$

In particular, we obtain

$$(6.14) \quad \nu \|D\mathbf{u}\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \leq c \|\nabla \mu\|^2 \leq c \|\nabla \Delta \Phi\|^2 + c \|\Phi\|_{H^{-1}}^2.$$

Taking the product of (6.13) with $-\Delta \Phi$ in H we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla \Phi\|^2 + \|\nabla \Delta \Phi\|^2 = -\langle \Delta[f(\phi) - f(\phi^*)], \Delta \Phi \rangle + \langle \mathbf{u} \cdot \nabla(\Phi + \phi^*), \Delta \Phi \rangle.$$

Observe that

$$\langle \Delta[f(\phi) - f(\phi^*)], -\Delta \Phi \rangle \leq \|\nabla[f(\phi) - f(\phi^*)]\| \|\nabla \Delta \Phi\| \leq c \|\nabla \Phi\|^2 + \frac{1}{4} \|\nabla \Delta \Phi\|^2.$$

The latter term can be estimated in light of (6.14) as

$$\begin{aligned}
\langle \mathbf{u} \cdot \nabla(\Phi + \phi^*), -\Delta \Phi \rangle &\leq \|\nabla \phi\|_{L^6} \|\mathbf{u}\| \|\Delta \Phi\|_{L^3} \\
&\leq c \|\mathbf{u}\| \|\Delta \Phi\|^{1/2} \|\nabla \Delta \Phi\|^{1/2} \\
&\leq c (\|\nabla \Delta \Phi\| + \|\Phi\|_{H^{-1}}) \|\nabla \Delta \Phi\|^{7/8} \|\Phi\|_{H^{-1}}^{1/8} \\
&\leq \frac{1}{4} \|\nabla \Delta \Phi\|^2 + c \|\Phi\|_{H^{-1}}^2.
\end{aligned}$$

We thus obtain, on account of (6.12), the inequality

$$\frac{1}{2} \frac{d}{dt} \|\nabla \Phi\|^2 + \frac{1}{2} \|\nabla \Delta \Phi\|^2 \leq c \|\Phi\|_{H^{-1}}^2 \leq \frac{c}{(1+t)^{2\theta/(1-2\theta)}}.$$

Recalling that $\|\nabla \Phi(t^*)\| \leq c$, an application of Lemma 2.14 yields (6.11). \square

6.2. Convergence of the velocity field \mathbf{u} . We are left to show that $\|\nabla \mathbf{u}(t)\|$ decays to 0 as $t \rightarrow \infty$. We will prove this by exploiting a further regularization of ϕ , namely,

Lemma 6.6. *The following inequality holds*

$$\|\nabla \mu(t)\| + \|\nabla \Delta \phi(t)\| \leq c_\nu, \quad \forall t \geq 2,$$

for some $c_\nu \rightarrow \infty$ as $\nu \rightarrow 0$.

Proof. Taking $w = \Delta^2 \mu$ in (2.7) we have

$$\langle \phi_t, \Delta^2 \mu \rangle = \langle \mathbf{u} \nabla \phi + \Delta \mu, \Delta^2 \mu \rangle = -\langle \nabla(\mathbf{u} \nabla \phi), \nabla \Delta \mu \rangle - \|\nabla \Delta \mu\|^2.$$

Exploiting the definition of μ , which gives $\mu_t = -\Delta \phi_t + f'(\phi) \phi_t$, we obtain

$$\begin{aligned} \langle \phi_t, \Delta^2 \mu \rangle &= \langle -\Delta \phi_t, -\Delta \mu \rangle \\ &= \langle \mu_t, -\Delta \mu \rangle - \langle f'(\phi) \phi_t, -\Delta \mu \rangle \\ &= \frac{1}{2} \frac{d}{dt} \|\nabla \mu\|^2 - \langle f'(\phi) \Delta \mu, -\Delta \mu \rangle + \langle f'(\phi)(\mathbf{u} \nabla \phi), -\Delta \mu \rangle. \end{aligned}$$

Hence we deduce that

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mu\|^2 + \|\nabla \Delta \mu\|^2 = -\langle f'(\phi) \Delta \mu, \Delta \mu \rangle + \langle f'(\phi)(\mathbf{u} \nabla \phi), \Delta \mu \rangle - \langle \nabla(\mathbf{u} \nabla \phi), \nabla \Delta \mu \rangle.$$

Let us estimate the terms on the right hand side. Bound (6.4) ensures

$$\|f'(\phi)\|_{L^\infty} \leq c_\nu,$$

where here and in the following c_ν is a generic constant depending on ν such that $c_\nu \rightarrow \infty$ as $\nu \rightarrow 0$. Then we have

$$\begin{aligned} -\langle f'(\phi) \Delta \mu, \Delta \mu \rangle &\leq \|f'(\phi)\|_{L^\infty} \|\Delta \mu\|^2 \leq c_\nu \|\Delta \mu\|^2 \\ &\leq c_\nu \|\nabla \Delta \mu\| \|\nabla \mu\| \leq \frac{1}{4} \|\nabla \Delta \mu\|^2 + c_\nu \|\nabla \mu\|^2, \end{aligned}$$

and

$$\begin{aligned} \langle f'(\phi)(\mathbf{u} \nabla \phi), \Delta \mu \rangle &\leq \|f'(\phi)\|_{L^\infty} \|\Delta \mu\| \|\mathbf{u}\|_{L^4} \|\nabla \phi\|_{L^4} \\ &\leq \|f'(\phi)\|_{L^\infty} \|\nabla \mu\|^{1/2} \|\nabla \Delta \mu\|^{1/2} \|\mathbf{u}\|^{1/4} \|\nabla \mathbf{u}\|^{3/4} \|\nabla \phi\|^{1/4} \|\nabla^2 \phi\|^{3/4} \\ &\leq \frac{1}{4} \|\nabla \Delta \mu\|^2 + c_\nu \|\nabla \mu\|^2 + c_\nu \|\nabla \mathbf{u}\|^2. \end{aligned}$$

Furthermore, by the Agmon inequality, we get

$$\begin{aligned} \langle \nabla(\mathbf{u} \nabla \phi), \nabla \Delta \mu \rangle &\leq \|\nabla \Delta \mu\| (\|\nabla \mathbf{u}\| \|\nabla \phi\|_{L^\infty} + \|\mathbf{u}\|_{L^6} \|\nabla^2 \phi\|_{L^3}) \\ &\leq c_\nu \|\nabla \Delta \mu\| \|\nabla \mathbf{u}\| \|\nabla \Delta \phi\|^{1/2} \\ &\leq \frac{1}{4} \|\nabla \Delta \mu\|^2 + c_\nu \|\nabla \mathbf{u}\|^2 \|\nabla \Delta \phi\|. \end{aligned}$$

Note that $\nabla \mu = -\nabla \Delta \phi + f'(\phi) \nabla \phi$. Then we infer

$$(6.15) \quad \|\nabla \Delta \phi\| \leq \|\nabla \mu\| + \|f'(\phi)\|_{L^\infty} \|\nabla \phi\| \leq (\|\nabla \mu\| + c_\nu),$$

so that

$$\langle \nabla(\mathbf{u} \nabla \phi), \nabla \Delta \mu \rangle \leq \frac{1}{4} \|\nabla \Delta \mu\|^2 + c_\nu \|\nabla \mathbf{u}\|^2 + c_\nu \|\nabla \mathbf{u}\|^2 \|\nabla \mu\|^2.$$

We thus end up with the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mu\|^2 + \frac{1}{4} \|\nabla \Delta \mu\|^2 \leq c_\nu (1 + \|\nabla \mathbf{u}\|^2 + \|\nabla \mu\|^2 + \|\nabla \mathbf{u}\|^2 \|\nabla \mu\|^2).$$

Recalling that $\int_0^\infty (\|\nabla \mathbf{u}(y)\|^2 + \|\nabla \mu\|^2) dy < c_\nu$ (see (3.1)), Lemma 2.13 yields

$$\|\nabla \mu(t+1)\|^2 \leq c_\nu, \quad t \geq 1.$$

Finally, by (6.15) we also have

$$\|\nabla \Delta \phi(t+1)\|^2 \leq c_\nu, \quad t \geq 1,$$

as claimed. \square

We are now ready to prove the convergence of \mathbf{u} to zero. To this aim, let us observe that, since μ^* is constant, then the equation for the velocity field \mathbf{u}^* associated with ϕ^* reduces to

$$\begin{cases} -\nu \Delta \mathbf{u}^* + \mathbf{u}^* = -\nabla p^* \\ \nabla \cdot \mathbf{u}^* = 0. \end{cases}$$

Therefore, upon multiplication by \mathbf{u}^* and integration over Ω , we deduce

$$\nu \|\nabla \mathbf{u}^*\|^2 + \|\mathbf{u}^*\|^2 = 0.$$

This implies $\mathbf{u}^* \equiv \mathbf{0}$, so that the following equation for the pressure p^* holds

$$\nabla p^* = \mu^* \nabla \phi^*.$$

Subtracting this last equation from (2.8) we deduce

$$-\nu \Delta \mathbf{u} + \mathbf{u} = -\nabla(p - p^*) + (\mu - \mu^*) \nabla(\Phi + \phi^*) + \mu^* \nabla \Phi.$$

Testing this relation by \mathbf{u} , owing to (6.4), we obtain

$$\begin{aligned} \nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2 &= \langle (\mu - \mu^*) \nabla(\Phi + \phi^*), \mathbf{u} \rangle + \langle \mu^* \nabla \Phi, \mathbf{u} \rangle \\ &\leq \|\mu - \mu^*\|_{L^3} \|\nabla(\Phi + \phi^*)\|_{L^6} \|\mathbf{u}\| + |\mu^*| \|\nabla \Phi\| \|\mathbf{u}\| \\ &\leq \|\mu - \mu^*\|^{1/2} \|\nabla(\mu - \mu^*)\|^{1/2} \|\phi\|_2 \|\mathbf{u}\| + c \|\nabla \Phi\| \|\mathbf{u}\| \\ &\leq \frac{1}{2} \|\mathbf{u}\|^2 + c \|\Delta \Phi\|^2 + c \|\mu - \mu^*\| \|\nabla \mu\|. \end{aligned}$$

Since

$$\|\mu - \mu^*\| \leq \|\Delta \Phi\| + \|f(\phi) - f(\phi^*)\| \leq c \|\Delta \Phi\|,$$

we obtain the estimate

$$\nu \|\nabla \mathbf{u}\|^2 + \frac{1}{2} \|\mathbf{u}\|^2 \leq c \|\Delta \Phi\| (\|\Delta \Phi\| + \|\nabla \mu\|).$$

By exploiting the boundedness of $\nabla \mu$ and $\Delta \Phi$, this yields

$$\|\mathbf{u}\|_1^2 \leq c_\nu \|\Delta \Phi\|,$$

for every $t \geq 2$. Finally, by interpolation and invoking the boundedness of $\|\nabla \Delta \phi\|$, we have

$$\|\mathbf{u}\|_1^2 \leq c_\nu \|\nabla \Phi\|^{1/2} \|\nabla \Delta \Phi\|^{1/2} \leq c_\nu \|\nabla \Phi\|^{1/2} = c_\nu \|\nabla(\phi - \phi^*)\|^{1/2}.$$

Therefore, Proposition 6.5 entails (2.13).

Remark 6.7. All the results and the estimates performed in this section and in Section 5 can be carried out in the case $\eta = 0$ with minor changes.

7. THE LIMIT $\nu \rightarrow 0$

Before studying the convergence of solutions to CHB system as $\nu \rightarrow 0$, we recall the following compactness result (see, e.g., [18]).

Theorem 7.1. *Let $X_0 \subset\subset X \subset X_1$ be three reflexive Banach spaces. Let $1 < a, b < \infty$ and define*

$$W^{a,b}(0, T; X_0, X_1) = \{z \in L^a(0, T; X_0) : \partial_t z \in L^b(0, T; X_1)\}.$$

Then $W^{a,b}(0, T; X_0, X_1)$ is reflexive and

$$W^{a,b}(0, T; X_0, X_1) \hookrightarrow L^a(0, T; X)$$

with compact embedding.

7.1. Proof of Theorem 2.11. Let $\phi_0 \in H^1$ and let $\{\nu_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\nu_n \rightarrow 0$ as $n \rightarrow \infty$. Consider the sequence $(\phi_{\nu_n}, \mathbf{u}_{\nu_n})$ of weak solutions corresponding to the CHB system with $\nu = \nu_n$. From the previous sections we know that the following bounds on $\{\phi_{\nu_n}\}_{n \in \mathbb{N}}$, $\{\mathbf{u}_{\nu_n}\}_{n \in \mathbb{N}}$ and $\{\mu_{\nu_n}\}_{n \in \mathbb{N}}$ are independent of n :

$$\begin{aligned} \|\phi_{\nu_n}\|_{L^\infty(0, T; H^1)} + \|\phi_{\nu_n}\|_{L^2(0, T; H^3)} &\leq c, \\ \|\mu_{\nu_n}\|_{L^2(0, T; H^1)} &\leq c, \\ \|\nabla \cdot (\phi_{\nu_n} \mathbf{u}_{\nu_n})\|_{L^{8/5}(0, T; H^{-1})} + \|\partial_t \phi_{\nu_n}\|_{L^{8/5}(0, T; H^{-1})} &\leq c, \\ \|\mathbf{u}_{\nu_n}\|_{L^2(0, T; \mathbf{H})} &\leq c, \\ \|\phi_{\nu_n} \nabla \mu_{\nu_n}\|_{L^{8/5}(0, T; \mathbf{H})} &\leq c. \end{aligned}$$

Thus deduce that there exists a relabeled sequence $\{\nu_n\}_{n \in \mathbb{N}}$ such that

$$\begin{aligned} \phi_{\nu_n} &\rightarrow \phi \quad \text{weakly in } L^2(0, T; H^3), \\ \mu_{\nu_n} &\rightarrow z \quad \text{weakly in } L^2(0, T; H^1), \\ \mathbf{u}_{\nu_n} &\rightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; \mathbf{H}). \end{aligned}$$

By the boundedness of $\partial_t \phi_{\nu_n}$ in $L^{8/5}(0, T; H^{-1})$ and by the uniqueness of L^p and distributional limits, we also have

$$\partial_t \phi_{\nu_n} \rightarrow \partial_t \phi \quad \text{weakly in } L^{8/5}(0, T; H^{-1}).$$

Applying Theorem 7.1 to ϕ_{ν_n} with $X_1 = H^{-1}$ and $X_0 = H^3$, up to a further subsequence, which will be relabeled ν_n , one has

$$\phi_{\nu_n} \rightarrow \phi \quad \text{strongly in } L^2(0, T; H^s),$$

for all $0 \leq s < 3$ and

$$\phi_{\nu_n} \rightarrow \phi \quad \text{a.e. in } \Omega \times (0, T).$$

Moreover, from the regularity of the potential f , it follows that $z = -\Delta \phi + f(\phi) = \mu$.

We can now consider the nonlinear terms appearing in (2.7) and (2.8). Let h be a positive real number. First of all, we show convergence of $\phi_{\nu_n} \nabla \mu_{\nu_n}$ to $\phi \nabla \mu$ in the following (weak) sense

$$\int_t^{t+h} \langle \phi_{\nu_n} \nabla \mu_{\nu_n} - \phi \nabla \mu, \mathbf{v} \rangle dt \rightarrow 0, \quad \forall \mathbf{v} \in \mathbf{V}.$$

The integrand can be rewritten as

$$\langle (\phi_{\nu_n} - \phi) \nabla \mu_{\nu_n}, \mathbf{v} \rangle + \langle \phi [\nabla \mu_{\nu_n} - \nabla \mu], \mathbf{v} \rangle.$$

The first term in this expression is bounded by

$$\langle (\phi_{\nu_n} - \phi) \nabla \mu_{\nu_n}, \mathbf{v} \rangle \leq \|\phi_{\nu_n} - \phi\|_{L^3} \|\nabla \mu_{\nu_n}\| \|\mathbf{v}\|_{L^6},$$

so that

$$\left| \int_t^{t+h} \langle (\phi_{\nu_n} - \phi) \nabla \mu_{\nu_n}, \mathbf{v} \rangle dt \right| \leq \|\phi_{\nu_n} - \phi\|_{L^2(0,T;L^3)} \|\mu_{\nu_n}\|_{L^2(0,T;H^1)} \|\mathbf{v}\|_{\mathbf{V}} \rightarrow 0.$$

Recalling that $\phi \in L^2(0, T; L^\infty(\Omega))$ the weak convergence of μ_{ν_n} in $L^2(0, T; H^1)$ implies

$$\langle \phi [\nabla \mu_{\nu_n} - \nabla \mu], \mathbf{v} \rangle \rightarrow 0.$$

Similarly we can deal with the convergence in $\nabla \cdot (\phi_{\nu_n} \mathbf{u}_{\nu_n})$. Indeed, we have

$$\int_t^{t+h} \langle \phi_{\nu_n} \mathbf{u}_{\nu_n} - \phi \mathbf{u}, \nabla v \rangle dt \rightarrow 0, \quad \forall v \in H^1.$$

This can be easily seen by rewriting the integrand as

$$\langle (\phi_{\nu_n} - \phi) \mathbf{u}_{\nu_n}, \nabla v \rangle + \langle \phi [\mathbf{u}_{\nu_n} - \mathbf{u}], \nabla v \rangle.$$

Indeed, the second term vanishes as $n \rightarrow \infty$ in light of the convergence

$$\mathbf{u}_{\nu_n} \rightarrow \mathbf{u} \quad \text{weakly in } L^2(0, T; \mathbf{H}),$$

and recalling the bound $\phi \in L^2(0, T; H^2) \subset L^2(0, T; L^\infty(\Omega))$, which yields in particular $\phi \nabla v \in L^2(0, T; (H)^3)$. Concerning the former, we observe

$$\begin{aligned} \left| \int_t^{t+h} \langle [\phi_{\nu_n} - \phi] \mathbf{u}_{\nu_n}, \nabla v \rangle dt \right| &\leq \int_t^{t+h} \|\nabla v\| \|\mathbf{u}_{\nu_n}\| \|\phi_{\nu_n} - \phi\|_{L^\infty} dt \\ &\leq \|\nabla v\| \left(\int_t^{t+h} \|\mathbf{u}_{\nu_n}\|^2 dt \right)^{1/2} \left(\int_t^{t+h} \|\phi_{\nu_n} - \phi\|_{L^\infty}^2 dt \right)^{1/2}. \end{aligned}$$

An application of Theorem 7.1 yields the compactness of $\{\phi_{\nu_n}\}$ in $L^2(0, T; L^\infty(\Omega))$, proving the required convergence.

Finally, let us consider the term involving the time derivative of ϕ . In particular, recalling that v is constant in time, we have

$$\int_t^{t+h} \partial_t \phi v dt = (\phi(t+h) - \phi(t))v.$$

Thanks to the boundedness of $\partial_t \phi$ in $L^{8/5}(0, T; H^{-1})$, the Lebesgue Theorem also gives

$$\frac{\phi(t+h) - \phi(t)}{h} \rightarrow \partial_t \phi(t) \quad \text{a.e. } t \in [0, T].$$

A repeated application of the Lebesgue Theorem implies that the couple (ϕ, \mathbf{u}) satisfies (2.7)-(2.8) for almost every time $t \in [0, T]$.

Moreover, observing that ϕ in $L^\infty(0, T; H^1)$ and $\phi \in \mathcal{C}([0, T]; H^{-1})$, it follows that ϕ is also weakly continuous taking values in H^1 .

Finally we show that

$$\lim_{t \rightarrow 0} \langle \phi(t), v \rangle = \langle \phi_0, v \rangle, \quad \text{for all } v \in H^{-1}.$$

Let $\psi: [0, T] \rightarrow \mathbb{R}$ be a \mathcal{C}^∞ function such that $\psi(0) = 1$ and $\psi(T) = 0$ and let $v \in H^1$ be arbitrary. Multiplying (2.7) with $\nu > 0$ by ψv and integrating over $\Omega \times [0, T]$ we obtain

$$-\int_0^T \langle \phi_{\nu_n}, \psi v \rangle dt + \int_0^T \langle \phi_{\nu_n} \mathbf{u}_{\nu_n}, \psi \nabla v \rangle dt + \int_0^T \langle \nabla \mu_{\nu_n}, \psi \nabla v \rangle dt = \langle \phi_0, v \rangle.$$

As before, we can pass to the limit as $\nu_n \rightarrow 0$, so obtaining

$$-\int_0^T \langle \phi, \psi v \rangle dt + \int_0^T \langle \phi \mathbf{u}, \psi \nabla v \rangle dt + \int_0^T \langle \nabla \mu, \psi \nabla v \rangle dt = \langle \phi_0, v \rangle.$$

Proceeding analogously in the case $\nu = 0$, we deduce

$$-\int_0^T \langle \phi, \psi v \rangle dt + \int_0^T \langle \phi \mathbf{u}, \psi \nabla v \rangle dt + \int_0^T \langle \nabla \mu, \psi \nabla v \rangle dt = \langle \phi(0), v \rangle.$$

Finally, a comparison between these last two equalities and the arbitrary choice of $v \in H^1$ gives $\phi(0) = \phi_0$.

8. THE CHB SYSTEM IN DIMENSION $N = 2$

In this section, we analyze the closeness between the solution to the CHB system and the solution to the CHHS system which are originated from regular initial data in H^2 .

Before proving our main result, i.e., Theorem 2.12, we derive some regularity estimates for the solutions of the CHB system in 2D which are uniform with respect to $\nu \geq 0$. Hence, from now on, let $\phi_0 \in H^2$ and denote by $c \geq 0$ a generic constant which may depend on $\|\phi_0\|_2$ but is *independent of* ν .

8.1. Higher-order bounds independent of ν . We shall exploit in a crucial way the following well-known inequalities which hold in dimension two:

$$(8.1) \quad \|f\|_{L^4}^2 \leq c(\|f\| \|\nabla f\| + \|f\|^2),$$

$$(8.2) \quad \|f\|_{L^4}^2 \leq c\|f\| \|\nabla f\|, \quad \text{if } \langle f \rangle = 0,$$

$$(8.3) \quad \|f\|_{L^\infty}^2 \leq c\|f\| \|f\|_{H^2}.$$

Proposition 8.1. *Let $\nu \geq 0$ be fixed and let $\phi(t) = S_\nu(t)\phi_0$. Then, the following estimate holds*

$$(8.4) \quad \|\phi(t)\|_2 + \int_t^{t+1} \|\phi(y)\|_4^2 dy \leq c, \quad \forall t \geq 0.$$

Furthermore, we have

$$(8.5) \quad \sup_{t \geq 0} \int_t^{t+1} (\|\mu(y)\|_2^2 + \|\mathbf{u}(y)\|_1^2) dy \leq c.$$

Proof. On account of (5.2) we find

$$\nu \|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2 = \langle \mu \nabla \phi, \mathbf{u} \rangle \leq \frac{1}{2} \|\mathbf{u}\|^2 + \frac{1}{2} \|\mu \nabla \phi\|^2,$$

which yields

$$\|\mathbf{u}\|^2 \leq \|\mu \nabla \phi\|^2.$$

Besides, by (8.1) and (8.2) we get

$$\|\mu \nabla \phi\|^2 \leq \|\mu\|_{L^4}^2 \|\nabla \phi\|_{L^4}^2 \leq c(\|\mu\| \|\nabla \mu\| + \|\mu\|^2) \|\nabla \phi\| \|\Delta \phi\|.$$

Since standard computations in light of (2.1) and (4.8) yield

$$\begin{aligned} \|\mu\| &\leq \|\Delta \phi\| + \|f(\phi)\| \leq c(1 + \|\Delta \phi\|), \\ \|\nabla \mu\| &\leq \|\nabla \Delta \phi\| + \|\nabla f(\phi)\| \leq c(1 + \|\nabla \Delta \phi\|), \end{aligned}$$

we end up with

$$(8.6) \quad \|\mathbf{u}\|^2 \leq c(1 + \|\Delta \phi\|^2) \|\nabla \Delta \phi\|.$$

By taking $w = \Delta^2 \phi$ in (2.7) we obtain

$$\frac{1}{2} \frac{d}{dt} \|\Delta \phi\|^2 + \|\Delta^2 \phi\|^2 = \langle \Delta f(\phi), \Delta^2 \phi \rangle + \langle \mathbf{u} \cdot \nabla \phi, \Delta^2 \phi \rangle.$$

We estimate the first term on the right hand side as follows

$$\begin{aligned} \langle \Delta f(\phi), \Delta^2 \phi \rangle &\leq \frac{1}{4} \|\Delta^2 \phi\|^2 + c \|\Delta f(\phi)\|^2 \\ &\leq \frac{1}{4} \|\Delta^2 \phi\|^2 + c(1 + \|\Delta \phi\|^2) \|\Delta \phi\|^2, \end{aligned}$$

where we exploit the 2D analog of (5.4) to control $\|\Delta f(\phi)\|$. Then we handle the remaining term as

$$\langle \mathbf{u} \cdot \nabla \phi, \Delta^2 \phi \rangle \leq c \|\mathbf{u} \cdot \nabla \phi\| \|\Delta^2 \phi\| \leq \frac{1}{4} \|\Delta^2 \phi\|^2 + c \|\mathbf{u} \cdot \nabla \phi\|^2.$$

Owing to (8.6) and the Agmon inequality (8.3), we infer

$$\begin{aligned} \|\mathbf{u} \cdot \nabla \phi\|^2 &\leq \|\mathbf{u}\|^2 \|\nabla \phi\|_{L^\infty}^2 \\ &\leq \|\mathbf{u}\|^2 \|\nabla \phi\| \|\nabla \Delta \phi\| \\ &\leq c(1 + \|\Delta \phi\|^2) \|\nabla \Delta \phi\|^2 \\ &\leq c \|\Delta \phi\|^2 \|\nabla \Delta \phi\|^2 + c \|\Delta \phi\| \|\Delta^2 \phi\| \\ &\leq c(1 + \|\nabla \Delta \phi\|^2) \|\Delta \phi\|^2 + \frac{1}{4} \|\Delta^2 \phi\|^2. \end{aligned}$$

Thus we obtain the differential inequality

$$(8.7) \quad \frac{1}{2} \frac{d}{dt} \|\Delta \phi\|^2 + \frac{1}{2} \|\Delta^2 \phi\|^2 \leq g(t) \|\Delta \phi\|^2,$$

where, in light of (3.4), $g(t) := c(1 + \|\Delta \phi(t)\|^2 + \|\nabla \Delta \phi(t)\|^2)$ satisfies

$$\sup_{t \geq 0} \int_t^{t+1} g(y) dy \leq c.$$

We can thus apply Lemma 2.13, so obtaining

$$\|\Delta\phi(t)\|^2 \leq c, \quad \forall t \geq 1.$$

In order to prove the required estimate for $t \in [0, 1]$ it is sufficient to apply the usual Gronwall lemma on $[0, t]$ to the inequality

$$\frac{d}{dt} \|\Delta\phi\|^2 \leq 2g(t) \|\Delta\phi\|^2.$$

Indeed this yields

$$\|\Delta\phi(t)\|^2 \leq \|\Delta\phi(0)\|^2 e^{2G(t)},$$

where

$$G(t) = \int_0^t g(y) dy \leq \int_0^1 g(y) dy \leq c, \quad \forall t \in [0, 1].$$

Hence we have

$$\|\Delta\phi(t)\|^2 \leq c, \quad \forall t \in [0, 1].$$

On account of this bound, a final integration of (8.7) on $[t, t+1]$ concludes the proof of (8.4).

In order to show the validity of (8.5), note that, by estimating again $\|\Delta f(\phi)\|$ as in (5.4), we get

$$\begin{aligned} \|\mu\|_2^2 &\leq c(\|\mu\|^2 + \|\Delta\mu\|^2) \\ &\leq c(\|f(\phi)\|^2 + \|\Delta\phi\|^2 + \|\Delta f(\phi)\|^2 + \|\Delta^2\phi\|^2) \\ &\leq c(1 + \|\Delta\phi\|^2 + \|\Delta\phi\|^4 + \|\Delta^2\phi\|^2), \end{aligned}$$

which, in light of (8.4), implies the integrability of μ . Concerning \mathbf{u} , by taking in (2.8) $\mathbf{v} = -\mathfrak{P}\Delta\mathbf{u}$ (\mathfrak{P} being the Leray projector), exploiting the Agmon inequality and the uniform H^2 -estimate for ϕ , we get

$$\begin{aligned} \|\mathbf{u}\|_1^2 &\leq \|\mu\nabla\phi\|^2 + \|\nabla\mu\nabla\phi\|^2 + \|\mu\nabla^2\phi\|^2 \\ &\leq \|\mu\|_{L^\infty}^2 \|\nabla\phi\|^2 + \|\nabla\mu\|_{L^4}^2 \|\nabla\phi\|_{L^4}^2 + c\|\mu\|_{L^\infty}^2 \|\Delta\phi\|^2 \\ &\leq c(1 + \|\mu\|_2^2). \end{aligned}$$

This implies the required integrability for $\|\mathbf{u}\|_1$. □

8.2. Proof of Theorem 2.12.

Proof. Let $\phi_0^\nu, \phi_0 \in H^2$ such that $\langle \phi_0^\nu \rangle = \langle \phi_0 \rangle$. Then denote by c a generic positive constant depending on R , where

$$R := \sup_{\nu > 0} \{\|\phi_0^\nu\|_2, \|\phi_0\|_2\} < \infty.$$

Let $(\phi_\nu, \mathbf{u}_\nu)$ be the weak solution to the CHB system with $\nu > 0$ originating from ϕ_0^ν , and (ϕ, \mathbf{u}) the solution to the CHHS system with initial datum ϕ_0 . Note that the difference $\phi = \phi_\nu - \phi$, $\bar{\mathbf{u}} = \mathbf{u}_\nu - \mathbf{u}$ is a weak solution to

$$(8.8) \quad \partial_t \bar{\phi} + \nabla \cdot (\phi_\nu \bar{\mathbf{u}}) + \nabla \cdot (\bar{\phi} \mathbf{u}) - \Delta \bar{\mu} = 0,$$

$$(8.9) \quad \bar{\mathbf{u}} = \nabla \bar{p} - \phi_\nu \nabla \bar{\mu} - \bar{\phi} \nabla \mu + \nu \Delta \mathbf{u}_\nu,$$

$$(8.10) \quad \nabla \cdot \bar{\mathbf{u}} = 0,$$

where

$$\bar{\mu} = -\Delta\bar{\phi} + [f(\phi_\nu) - f(\phi)],$$

and $\langle\bar{\phi}\rangle = 0$.

Taking $-\Delta\bar{\phi}$ as test function in the weak formulation of (8.8), we obtain

$$\frac{d}{dt} \frac{1}{2} \|\nabla\bar{\phi}\|^2 + \langle\phi_\nu\bar{\mathbf{u}}, \nabla\Delta\bar{\phi}\rangle + \langle\bar{\phi}\mathbf{u}, \nabla\Delta\bar{\phi}\rangle + \langle\nabla\bar{\mu}, \nabla\Delta\bar{\phi}\rangle = 0.$$

On the other hand, we have

$$\langle\nabla\bar{\mu}, \nabla\Delta\bar{\phi}\rangle = -\|\nabla\Delta\bar{\phi}\|^2 + \langle\nabla[f(\phi_\nu) - f(\phi)], \nabla\Delta\bar{\phi}\rangle,$$

so that

$$(8.11) \quad \frac{d}{dt} \frac{1}{2} \|\nabla\bar{\phi}\|^2 + \|\nabla\Delta\bar{\phi}\|^2 = -\langle\phi_\nu\bar{\mathbf{u}}, \nabla\Delta\bar{\phi}\rangle - \langle\bar{\phi}\mathbf{u}, \nabla\Delta\bar{\phi}\rangle + \langle\nabla[f(\phi_\nu) - f(\phi)], \nabla\Delta\bar{\phi}\rangle.$$

Let us now take $\bar{\mathbf{u}}$ in the weak formulation of (8.9). This gives

$$(8.12) \quad \|\bar{\mathbf{u}}\|^2 = -\langle\phi_\nu\nabla\bar{\mu}, \bar{\mathbf{u}}\rangle - \langle\bar{\phi}\nabla\mu, \bar{\mathbf{u}}\rangle - \nu\langle\nabla\mathbf{u}_\nu, \nabla\bar{\mathbf{u}}\rangle.$$

Note that, by definition of $\bar{\mu}$, there holds

$$-\langle\phi_\nu\nabla\bar{\mu}, \bar{\mathbf{u}}\rangle = \langle\phi_\nu\nabla\Delta\bar{\phi}, \bar{\mathbf{u}}\rangle - \langle\phi_\nu\nabla[f(\phi_\nu) - f(\phi)], \bar{\mathbf{u}}\rangle.$$

Hence, adding (8.11) with (8.12) we end up with

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\nabla\bar{\phi}\|^2 + \|\nabla\Delta\bar{\phi}\|^2 + \|\bar{\mathbf{u}}\|^2 &= -\nu\langle\nabla\mathbf{u}_\nu, \nabla\bar{\mathbf{u}}\rangle \\ &\quad - \langle\bar{\phi}\mathbf{u}, \nabla\Delta\bar{\phi}\rangle - \langle\bar{\phi}\nabla\mu, \bar{\mathbf{u}}\rangle - \langle\phi_\nu\nabla[f(\phi_\nu) - f(\phi)], \bar{\mathbf{u}}\rangle + \langle\nabla[f(\phi_\nu) - f(\phi)], \nabla\Delta\bar{\phi}\rangle. \end{aligned}$$

We now estimate the terms on the right hand side. First of all, we have

$$-\langle\bar{\phi}\mathbf{u}, \nabla\Delta\bar{\phi}\rangle \leq c\|\bar{\phi}\|_1\|\mathbf{u}\|_1\|\nabla\Delta\bar{\phi}\| \leq \frac{1}{4}\|\nabla\Delta\bar{\phi}\|^2 + c\|\mathbf{u}\|_1^2\|\bar{\phi}\|_1^2.$$

Besides, the following inequality holds

$$-\langle\bar{\phi}\nabla\mu, \bar{\mathbf{u}}\rangle \leq \|\bar{\phi}\|_1\|\bar{\mathbf{u}}\|\|\nabla\mu\|_{L^3} \leq \frac{1}{2}\|\bar{\mathbf{u}}\|^2 + c\|\Delta\mu\|^2\|\bar{\phi}\|_1^2.$$

We are left to deal with the term

$$\begin{aligned} \langle\nabla[f(\phi_\nu) - f(\phi)], \nabla\Delta\bar{\phi}\rangle &\leq \|\nabla[f(\phi_\nu) - f(\phi)]\| \|\nabla\Delta\bar{\phi}\| \\ &\leq \frac{1}{4}\|\nabla\Delta\bar{\phi}\|^2 + c\|\nabla[f(\phi_\nu) - f(\phi)]\|^2, \end{aligned}$$

where

$$\|\nabla[f(\phi_\nu) - f(\phi)]\|^2 \leq \|[f'(\phi_\nu) - f'(\phi)]\nabla\phi_\nu\|^2 + \|f'(\phi)\nabla\bar{\phi}\|^2.$$

By exploiting the uniform H^2 -estimates both for ϕ_ν and ϕ obtained in (8.4) and condition (2.3), we have

$$\begin{aligned} \|f'(\phi)\nabla\bar{\phi}\|^2 &= \int_{\Omega} |f'(\phi)\nabla\bar{\phi}|^2 \leq \|f'(\phi)\|_{L^\infty}^2 \|\nabla\bar{\phi}\|^2 \\ &\leq c(1 + \|\phi\|_{L^\infty}^4) \|\nabla\bar{\phi}\|^2 \\ &\leq c\|\bar{\phi}\|_1^2, \end{aligned}$$

and, analogously,

$$\begin{aligned} \| [f'(\phi_\nu) - f'(\phi)] \nabla \phi_\nu \|^2 &\leq c \int_{\Omega} |(1 + |\phi_\nu| + |\phi|) \bar{\phi} \nabla \phi_\nu|^2 \\ &\leq c \|\bar{\phi}\|_{L^4}^2 \|\nabla \phi_\nu\|_{L^4}^2 \\ &\leq c \|\bar{\phi}\|_1^2. \end{aligned}$$

Thus we have the control

$$\|\nabla [f(\phi_\nu) - f(\phi)]\|^2 \leq c \|\bar{\phi}\|_1^2.$$

Using again (8.4), the remaining term involving f can be treated in the following way:

$$\begin{aligned} -\langle \phi_\nu \nabla [f(\phi_\nu) - f(\phi)], \bar{\mathbf{u}} \rangle &\leq \|\phi_\nu\|_{L^\infty} \|\nabla [f(\phi_\nu) - f(\phi)]\| \|\bar{\mathbf{u}}\| \\ &\leq \frac{1}{4} \|\bar{\mathbf{u}}\|^2 + c \|\nabla [f(\phi_\nu) - f(\phi)]\|^2 \\ &\leq \frac{1}{4} \|\bar{\mathbf{u}}\|^2 + c \|\bar{\phi}\|_1^2. \end{aligned}$$

In addition, we have

$$\nu |\langle \nabla \mathbf{u}_\nu, \nabla \bar{\mathbf{u}} \rangle| \leq \nu (\|\nabla \mathbf{u}_\nu\|^2 + \|\bar{\mathbf{u}}\|_1^2).$$

Note that, recalling (8.5),

$$k(\cdot) := \|\nabla \mathbf{u}_\nu(\cdot)\|^2 + \|\bar{\mathbf{u}}(\cdot)\|_1^2 \in L^1(0, T),$$

for every $T > 0$, uniformly with respect to $\nu \geq 0$.

Collecting all the above inequalities, we end up with

$$\frac{d}{dt} \|\bar{\phi}\|_1^2 + \frac{1}{4} \|\bar{\mathbf{u}}\|^2 \leq h(t) \|\bar{\phi}\|_1^2 + \nu k(t),$$

where

$$h(t) = c(1 + \|\Delta \mu(t)\|^2 + \|\mathbf{u}(t)\|_1^2).$$

Thanks to (8.5) once more, $h \in L^1(0, T)$ uniformly with respect to $\nu \geq 0$. Therefore, an application of the Gronwall lemma provides, for all $t \in [0, T]$,

$$\|\phi_\nu(t) - \phi(t)\|_1^2 \leq \|\phi_0^\nu - \phi_0\|_1^2 e^{\int_0^t h(y) dy} + \nu \int_0^t k(y) dy,$$

which, in particular, entails that

$$\|\phi_\nu(t) - \phi(t)\|_1^2 \leq \|\phi_0^\nu - \phi_0\|_1^2 e^{C_T} + \nu C_T,$$

having set $C_T = \max\{\int_0^T h(y) dy, \int_0^T k(y) dy\} < \infty$. Integrating the differential inequality on $[0, t]$, $t \leq T$, up to enlarging C_T we also obtain

$$\int_0^t \|\mathbf{u}_\nu - \mathbf{u}\|^2 \leq \|\phi_0^\nu - \phi_0\|_1^2 e^{C_T} + \nu C_T. \quad \square$$

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